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A CRITCAL STUDY OF THE RADIATIVE TRANSFER APPROACH FOR THE REMOTE SENSING OF LAYERED RANDOM MEDIA

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14. ABSTRACT
The Radiative Transfer (RT) approach is widely used in applications involving scattering from layered random media with rough interfaces. Although this approach involves approximations, in most applications they are not explicitly stated or well understood. In order to better understand the RT approach to our problem we adopt a statistical wave approach and then transition to the RT equations. The geometry of our problem consists of a multi-layer discrete random medium with rough boundaries which are planar on the average. The random medium in each layer consists of a homogeneous background medium in which discrete scatterers are randomly distributed. The regions above and below the random medium stack are homogeneous. Using the Greens functions of the problem without the volumetric fluctuations we represent our problem as a system of integral equations. Employing the T-matrix description we first average with respect to volumetric fluctuations and then use the Twersky approximation to obtain a system of integral equations. We next average with respect to surface fluctuations, apply the weak surface correlation approximation and arrive at a closed system of integral equations for the first and second moments of the electric fields. We use the Wigner transforms to translate the coherence functions to radiant intensities, which are the fundamental quantities in the RT approach. On applying the quasi-static field approximation we hence arrive at a system of equations identical to those used in the RT approach. From this study we learn that there are more conditions involved in the RT approach than widely believed to be sufficient. This means that the applicability of the RT approach is much smaller than what is generally conceived by the remote sensing community.

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1. Summary

The Radiative Transfer (RT) approach is widely used in applications involving scattering from layered random media with rough interfaces (F.T. Ulaby, R.K. Moore and AK Fung, *Microwave Remote Sensing*, Vol. 3, Artech House, 1986). Although this approach involves approximations, in most applications they are not explicitly stated or well understood. The RT approach for random media with non-scattering boundaries has been well studied (M.I. Mishchenko, *Appl. Optics*, 41, 7114-34, 2002). In contrast our problem has scattering boundaries which are randomly rough. In order to better understand the RT approach to our problem we adopt a statistical wave approach and then transition to the RT equations. The geometry of our problem consists of a multi-layer discrete random medium with rough boundaries which are planar on the average. The random medium in each layer consists of a homogeneous background medium in which discrete scatterers are randomly distributed. The statistical characteristics of the random medium in each layer are independent of each other and independent of the statistics describing the rough interfaces. The regions above and below the random medium stack are homogeneous. Using the Greens functions of the problem without the volumetric fluctuations we represent our problem as a system of integral equations. Employing the T-matrix description we first average with respect to volumetric fluctuations and then use the Twersky approximation to obtain a system of integral equations. We next average with respect to surface fluctuations, apply the weak surface correlation approximation and arrive at a closed system of integral equations for the first and second moments of the electric fields. We use the Wigner transforms to translate the coherence functions to radiant intensities, which are the fundamental quantities in the RT approach. On applying the quasi-static field approximation we hence arrive at a system of equations identical to those used in the RT approach. From this study we learn that there are more conditions involved in the RT approach than widely believed to be sufficient. This means that the applicability of the RT approach is much smaller than what is generally conceived by the remote sensing community.

2. Introduction

The radiative transfer (RT) theory is widely used in remote sensing problems [16,28,17,15,1,38]. Often the model of layered random medium with rough interfaces is used. Multiple scattering processes in this structure are well represented by the RT equations. Although quite successful in numerous applications in various disciplines, it is known that the RT approach involves approximations. Often people in the remote sensing community are not quite familiar with the approximations involved in the RT approach and hence there has been inappropriate use of the RT approach in the literature. Since the phenomenological RT theory [7,35] was first developed for light scattering in planetary atmospheres the RT conditions prevalent in the atmospheric context has been popularly identified as sufficient conditions for employing the RT theory. However, we notice that the RT theory has been used for a variety of different problems [25,32,26,34] in various applications with complex geometries. It is not clear whether the classical conditions associated with the RT theory are sufficient in all situations. In this report we will review the approximations involved and clarify misconceptions. In order to better understand the RT approach we employ the more rigorous statistical wave theory to the problem and hence make the transition to the RT equations. In this process we clarify and explain the assumptions or approximations involved in the RT approach. By following this procedure we found that there are more conditions embedded in the RT approach than widely believed to be sufficient. For our study we have considered a multi-layer random medium composed of discrete scatterers (see Figure 1). By considering several special cases of this general problem we show that the number of conditions implied in the RT approach reduces with simpler geometries. Our conclusions are not just for the case discrete random media. Similar conditions apply for the random continuum as well. Similar conditions apply for the random continuum as well [27]. There, the volumetric fluctuations were modeled as a random continuum. The random continuum concept is rather abstract and difficult to model several problems encountered in applications. Perhaps the most serious drawback of the continuum model is that it often impossible to relate the key statistical parameters to observable quantities. The current report is devoted to the remote sensing applications. A discrete scatterer model is most appropriate and realistic for modeling real world problems [24]. The analysis used in the discrete random media is quite different from the continuum case. Furthermore, the conditions and assumptions embedded in the RT approach are also different in the discrete model case.

The report is organized as follows. First we describe the geometry of the problem. Next we give the radiative transfer approach to the problem. The next section is on the statistical wave approach to the problem. This occupies the major part of the report. It deals with the derivation and analysis of the first and second moments of the wave functions. A transition is next made to RT equations. Next we enumerate and discuss the conditions implied in the RT approach.

3. Methods, Assumptions, and Procedures

3.1 Description of the Problem

The geometry of the problem is shown in Figure 1. We have an *N*-layer random medium stack with rough interfaces which, on the average, are parallel planes. Let ε_j be the permittivity of the background medium, and let ε_{js} be the permittivity of the scatterers in the *j*-th layer. The location and orientation of the scatterers are random functions characterizing the fluctuations. On the average the problem is translationally invariant and isotropic in the azimuth. We assume that the volumetric fluctuations in each layer are statistically independent of each other. Let N_j be the number of scatterers, and let ρ_j be the density of the scatterers (number of scatterers per unit volume) of the *j*-the layer. The permeability of all the layers is that of free space. The randomly rough interfaces are given as $z = z_j + \zeta_j(\mathbf{r}_\perp)$. Then ζ_j are zero-mean isotropic stationary random processes independent of volumetric fluctuations of the problem. Let $z_0 = 0$, and let d_j be the thickness of the *j*-th layer. Let $z_0 = 0$, and let d_j be the thickness of the *j*-th layer. The media above and below the stack are homogeneous with parameters ε_j, k_0, and ε_{N+1}, k_{N+1}, respectively. This system is excited by a monochromatic electromagnetic plane wave and we are interested in formulating the resulting multiple scattering process.

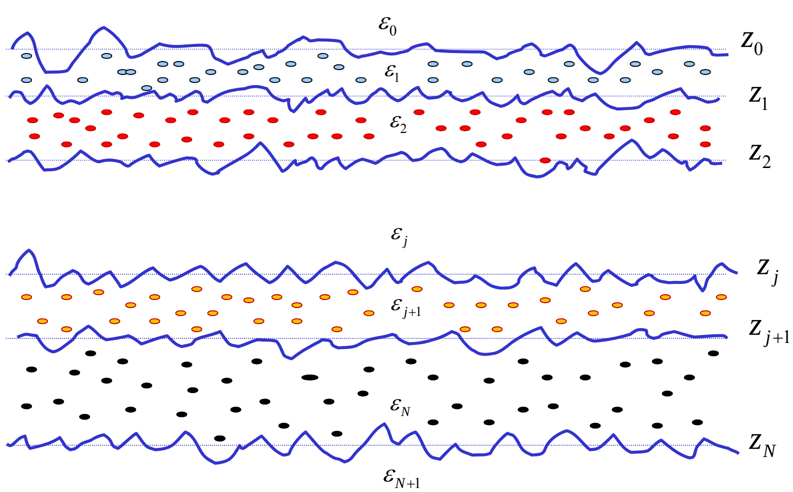

Figure 1. Geometry of the Problem

3.2 Radiative Transfer Approach

Multiple scattering in a complex environment is well described by the radiative transfer theory. This theory is not only conceptually simple but also very efficient. The fundamental quantity here is the specific intensity $\mathbf{I}$ which is governed by the following equation [7,35,11]

$$\hat{s}\cdot\nabla\mathbf{I}(\mathbf{r},\hat{s})+\bar{\bar{\gamma}}\mathbf{I}(\mathbf{r},\hat{s})=\int\bar{\bar{\mathbf{P}}}(\hat{s},\hat{s}')\mathbf{I}(\mathbf{r},\hat{s}')d\Omega' \tag{1}$$

One may regard this equation as a statement of conservation of energy density $\mathbf{I}$ which is a phase-space quantity at position $\mathbf{r}$ and direction $\hat{s}$. $\bar{\bar{\gamma}}$ is the extinction matrix which is a measure of loss of energy due to scattering in other directions. $\bar{\bar{\mathbf{P}}}$ is the phase matrix representing increase in energy density due to scattering from neighbouring elements. Ω is the solid angle subtended by $\hat{s}$. Given the statistical characteristics of the medium one can calculate the phase function using the single scattering theory for elements that constitute the random medium of the layer [39,12,6,23]. The extinction matrix is hence calculated using the optical theorem. The specific intensity in each layer is governed by an equation similar to (1). Since our layer problem has translational invariance in azimuth the RT equation for the *m*-th layer takes the following form,

$$\cos\theta\frac{d}{dz}\mathbf{I}_m(z,\hat{s})+\bar{\bar{\gamma}}_m\mathbf{I}_m(z,\hat{s})=\int_{\Omega_m}\int\bar{\bar{\mathbf{P}}}_m(\hat{s},\hat{s}')\mathbf{I}_m(z,\hat{s}')d\Omega' \tag{2}$$

where the subscript *m* denotes that the quantity corresponds to that of the *m*-th layer and θ is the elevation angle of $\hat{s}$. This set of RT equations is complemented by a set of boundary conditions which is in turn based on energy conservation considerations. To be more precise, we impose the condition that the energy flux density at each interface is conserved. This leads to the following boundary conditions on the *m*-th interface

$$\mathbf{I}_m^u(z_m,\hat{s})=\int\mathbf{R}_{\mathbf{m+1},m}(\hat{s},\hat{s}')\,\mathrm{I}_m^d(z_{m,}\hat{s}')d\Omega'+\int\mathbf{X}_{m,\mathbf{m+1}}(\hat{s},\hat{s}')\,\mathrm{I}_{m+1}^u(z_{m,}\hat{s}')d\Omega' \tag{3}$$

The boundary conditions on the $(m-1)$-th interface are given as

$$\mathbf{I}_m^d(z_{m-1},\hat{s})=\int\bar{\mathbf{R}}_{\mathbf{m-1},m}(\hat{s},\hat{s}')\,\mathrm{I}_m^u(z_{m-1,}\hat{s}')d\Omega'+\int\bar{\mathbf{X}}_{m,\mathbf{m-1}}(\hat{s},\hat{s}')\,\mathrm{I}_{m-1}^d(z_{m-1,}\hat{s}')d\Omega' \tag{4}$$

where $\overline{\mathbf{R}}_{mn}$ and $\overline{\mathbf{X}}_{mn}$ are the local reflection and transmission Mueller matrices. To be more specific, $\overline{\mathbf{R}}_{mn}$ represents the reflection matrix of waves incident from medium *n* on the interface separating medium *m* and medium *n*. The superscripts *u* and *d* indicate whether the intensity corresponds to a wave travelling upwards or downwards. These Mueller matrices are often calculated using some asymptotic theory such as the Kirchhoff approximation [41,5,38,37]. The integration in these expressions are over a solid angle (hemisphere) corresponding to $\hat{s}'$. Suppose we have a time-harmonic electromagnetic plane wave incident on this stack from above. Then the downward travelling intensity in Region 0 is

$$\mathbf{I}_0^d(z,\hat{s}) = \mathbf{B}_0 \delta(\cos\theta_0 - \cos\theta_i)\delta(\phi_0 - \phi_i) \tag{5}$$

where $\mathbf{B}_0$ is the intensity of the incident plane wave and $\{\theta_i, \varphi_i\}$ describes its direction. Since there is no source or scatterer in Region *N*+1,

$$\mathbf{I}_{N+1}^u(z,\hat{s}) = 0$$

Notice again that these boundary conditions represent conservation of intensity at the interfaces. We should point out that the radiative transfer approach as applied to a particular problem is only a model based on certain assumptions. Since the RT theory is used in a variety of applications, the particular assumptions involved are described in terms of different terminologies, specific to the discipline where it is used. One good way to understand in more general terms the RT approach and the underlying assumptions is to compare it with the more rigorous wave approach. For the case of an unbounded random medium this kind of study was carried out in the 1970s [3,2]. From that study we learn that the radiative transfer theory can be applied under the following conditions:

1. Quasi-stationary field approximation
2. Sparse distribution
3. Statistical homogeneity of the medium fluctuations

These are the well-known conditions that we associate with the RT approach. However, our problem has bounded structures and, further, they are randomly rough. The question is this: are the above conditions sufficient to apply the RT approach for our problem? This is the motivation for this report. We follow the wave approach to this problem, derive the equations for the intensities, and hence make the transition to the RT equations. This procedure enables us to better understand the necessary conditions for using the RT approach for our problem.

3.3 Statistical Wave Approach

The following are the equations that govern the waves in the layer structure:

$$\nabla\times\nabla\times\mathbf{E}_j - k_j^2\mathbf{E}_j = v_j\mathbf{E}_j \qquad j = 1,\cdots,N \tag{6}$$

where

$$v_j \equiv \sum_{i=1}^{N_j} \mathrm{Q}_{\mathrm{ji}}(\mathbf{r}) \tag{7a}$$

$$Q_{ji}(\mathbf{r}) = \begin{cases} k_{js}^2 - k_j^2 & \mathbf{r}\in V_{ji} \\ 0 & \text{otherwise} \end{cases} \tag{7b}$$

$$k_{js}^2 = \omega^2\mu\varepsilon_{js} \qquad k_j^2 = \omega^2\mu\varepsilon_j \tag{7c}$$

V_{ji} is the domain of the *i*-th scatterer and N_j is the number of scatterers in layer *j*. Thus v_j represents the volumetric fluctuations in Region *j*. For the homogeneous regions above and below we have

$$\nabla\times\nabla\times\mathbf{E}_0 - k_0^2\mathbf{E}_0 = 0 \tag{8a}$$

$$\nabla\times\nabla\times\mathbf{E}_{N+1} - k_{N+1}^2\mathbf{E}_{N+1} = 0 \tag{8b}$$

The boundary conditions at the *j*-th interface are

$$\hat{\mathbf{n}}\times\mathbf{E}_j(\mathbf{r}_\perp,\zeta_j) = \hat{\mathbf{n}}\times\mathbf{E}_{j+1}(\mathbf{r}_\perp,\zeta_j) \tag{9a}$$

$$\hat{\mathbf{n}}\times\nabla\times\mathbf{E}_j(\mathbf{r}_\perp,\zeta_j) = \hat{\mathbf{n}}\times\nabla\times\mathbf{E}_{j+1}(\mathbf{r}_\perp,\zeta_j) \tag{9b}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the *j*-th interface with normal pointing into the medium *j*. This system is complemented by the radiation conditions well away from the stack. We assume that we know the solution to the problem without volumetric fluctuations, and denote it as $\breve{\mathbf{E}}$. Let the Green's functions to this problem be denoted as $\breve{\bar{\mathbf{G}}}_{ij}$. These Green's functions are governed by the following set of equations:

$$\nabla\times\nabla\times\breve{\bar{\mathbf{G}}}_{jk}(\mathbf{r},\mathbf{r}') - k_j^2\breve{\bar{\mathbf{G}}}_{jk}(\mathbf{r},\mathbf{r}') = \bar{\mathbf{I}}\delta_{jk}\delta(\mathbf{r}-\mathbf{r}') \tag{10a}$$

$$\hat{\mathbf{n}} \times \breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}') = \hat{\mathbf{n}} \times \breve{\bar{\bar{\mathbf{G}}}}_{(j+1)k}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}') \tag{10b}$$

$$\hat{\mathbf{n}} \times \nabla \times \breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}') = \hat{\mathbf{n}} \times \nabla \times \breve{\bar{\bar{\mathbf{G}}}}_{(j+1)k}(\mathbf{r}_\perp, \zeta_j; \mathbf{r}') \tag{10c}$$

where $\bar{\mathbf{I}}$ is unit dyad. The boundary conditions above are for the *j*-th interface. Similarly on the (*j*-1)-th interface we have the following boundary conditions.

$$\hat{\mathbf{n}} \times \breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r}_\perp, \zeta_{j-1}; \mathbf{r}') = \hat{\mathbf{n}} \times \breve{\bar{\bar{\mathbf{G}}}}_{(j-1)k}(\mathbf{r}_\perp, \zeta_{j-1}; \mathbf{r}') \tag{11a}$$

$$\hat{\mathbf{n}} \times \nabla \times \breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r}_\perp, \zeta_{j-1}; \mathbf{r}') = \hat{\mathbf{n}} \times \nabla \times \breve{\bar{\bar{\mathbf{G}}}}_{(j-1)k}(\mathbf{r}_\perp, \zeta_{j-1}; \mathbf{r}') \tag{11b}$$

Using these Green's functions and the radiation conditions the wave functions can be represented as

$$\mathbf{E}_j(\mathbf{r}) = \breve{\mathbf{E}}_j(\mathbf{r}) + \sum_{k=1}^{N} \int_{\Omega_k} \breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r}, \mathbf{r}') v_k(\mathbf{r}') \mathbf{E}_k(\mathbf{r}') d\mathbf{r}' \qquad j = 0, 1, \ldots, N+1 \tag{12}$$

where $\Omega_k = \{\mathbf{r}'_\perp; \zeta_k < z' < \zeta_{k-1}\}$. Note that $v_0 = v_{N+1} = 0$. In order to carry out multiple scattering analysis with a distribution of discrete scatterers it is convenient to employ the concept of transition operator $\bar{\mathbf{T}}$ [15,42,40]. Suppose we know the electric field $\mathbf{E}^l$ incident on the *l*-th scatterer. We introduce the transition operator $\bar{\mathbf{T}}^l$ such that the electric field scattered by the *l*-th scatterer is given as $\bar{\mathbf{T}}^l \mathbf{E}^l$. Using this concept (12) may be expressed as

$$\mathbf{E}_j(\mathbf{r}) = \breve{\mathbf{E}}_j(\mathbf{r}) + \sum_{k=1}^{N} \sum_{l=1}^{N_k} \int_{\Omega_k} d\mathbf{r}' \int_{\Omega_k} d\mathbf{r}'' \breve{\bar{\bar{\mathbf{G}}}}_{jk}(\mathbf{r}, \mathbf{r}') \bar{\mathbf{T}}_k^l(\mathbf{r}', \mathbf{r}'') \mathbf{E}_k^l(\mathbf{r}'') \qquad j = 0, 1, \ldots, N+1 \tag{13}$$

Note that $\bar{\mathbf{T}}_k^l$ depends only on the *l*-th scatterer. To proceed further with the multiple scattering analysis it is expedient to use symbolic representation of (13).

$$\mathbf{E}_j = \breve{\mathbf{E}}_j + \breve{\bar{\bar{\mathbf{G}}}}_{jk} \bar{\mathbf{T}}_k^l \mathbf{E}_k^l \tag{14}$$

First we average (14) w.r.t. volumetric fluctuations to get

$$\left\langle \mathbf{E}_j \right\rangle_v = \breve{\mathbf{E}}_j + \breve{\overline{\overline{\mathbf{G}}}}_{jk} \left\langle \overline{\mathbf{T}}_k^l \mathbf{E}_k^l \right\rangle_v \tag{15}$$

where the subscript v denotes volumetric averaging. Since there are N_k scatterers in the k-th layer

$$\left\langle \overline{\mathbf{T}}_k^l \mathbf{E}_k^l \right\rangle_v = \int \overline{\mathbf{T}}_k^l \mathbf{E}_k^l \, p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}; \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_k}) d\mathbf{r} d\mathbf{s} \tag{16}$$

where p is the joint probability density function of finding the scatterers at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}$ with orientations $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_k}$. We assume that the positions and orientations are independent of each other. In other words

$$p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}; \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_k}) = p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}) \, p(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_k}) \tag{17}$$

Furthermore, assume that the orientation of the particle at position $\mathbf{r}_1$ is independent of the orientation of all other particles, which means

$$p(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_k}) = \sum_{j=1}^{N_k} p(\mathbf{s}_j) \tag{18}$$

We next express the joint position probability density function as

$$p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}) = p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_l', \ldots, \mathbf{r}_{N_k} \left| \mathbf{r}_l \right|) \, p(\mathbf{r}_l) \tag{19}$$

where $p(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_l', \ldots, \mathbf{r}_{N_k} \mid \mathbf{r}_l)$ is the conditional pdf of finding scatterers at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{N_k}$ by fixing the l-th scatterer is at $\mathbf{r}_l$. The prime in $\mathbf{r}_l'$ denotes that $\mathbf{r}_l$ should be omitted in the argument list of the conditional probability density function. Substituting this relation in (16) we obtain

$$\left\langle \overline{\mathbf{T}}_k^l \mathbf{E}_k^l \right\rangle = \left\langle \overline{\mathbf{T}}_k^l \left\langle \mathbf{E}_k^l \right\rangle^l \right\rangle \tag{20}$$

where $\left\langle \mathbf{E}_k^l \right\rangle^l$ denotes conditional average with scatterer l fixed at $\mathbf{r}_l$**.** If N_k is large and the distance between scatterers is large then we can approximate

$$\left\langle \mathbf{E}_k^l \right\rangle^l \approx \left\langle \mathbf{E}_k^l \right\rangle \tag{21}$$

This is called the Foldy's approximation and is applicable for sparse media. Under this approximation (15) becomes

$$\left\langle \mathbf{E}_j \right\rangle_v \cong \breve{\mathbf{E}}_j + \rho_k \breve{\overline{\mathbf{G}}}_{jk} \left\langle \overline{\mathbf{T}}_k \right\rangle \left\langle \mathbf{E}_k \right\rangle_v \tag{22}$$

where ρ_k is the density of the scatterers in layer *k*. Now operate the above equation by $\nabla \times \nabla \times \overline{\mathbf{I}} - k_j^2 \overline{\mathbf{I}}$ to obtain

$$\nabla \times \nabla \times \left\langle \mathbf{E}_k \right\rangle_v - k_j^2 \left\langle \mathbf{E}_k \right\rangle_v = \rho_j \left\langle \overline{\mathbf{T}}_j \right\rangle \left\langle \mathbf{E}_j \right\rangle_v \tag{23}$$

Next average this over surface fluctuations to get

$$\nabla \times \nabla \times \left\langle \mathbf{E}_j \right\rangle_{vs} - k_j^2 \left\langle \mathbf{E}_j \right\rangle_{vs} = \rho_j \left\langle \overline{\mathbf{T}}_j \right\rangle \left\langle \mathbf{E}_j \right\rangle_{vs} \tag{24}$$

Note that the transition operators are independent of surface fluctuations. From this we see that

$$\nabla \times \nabla \times \left\langle \mathbf{E}_j \right\rangle_{vs} - k_j^2 \left\langle \mathbf{E}_j \right\rangle_{vs} = 0 \qquad\qquad j = 0, N+1 \tag{25}$$

which means that the coherent propagation constants in the regions above and below the layered stack are unaffected by the fluctuations of the problem. However, they indeed get modified in the layered stack region. On writing (24) as

$$\left\{ \nabla \times \nabla \times \overline{\mathbf{I}} - k_j^2 - \rho_j \left\langle \overline{\mathbf{T}}_j \right\rangle \right\} \left\langle \mathbf{E}_j \right\rangle_{vs} = 0 \tag{26}$$

we infer that $\chi_j = \sqrt{k_j^{\,2} + \rho_j \left\langle \overline{\mathbf{T}}_j \right\rangle}$ represents the mean propagation constant, in operator form, for coherent waves in layer *j*.

Since the problem is statistically homogeneous in azimuth the mean fields in our system have the following form:

$$\begin{aligned} \left\langle \mathbf{E}_j^p(\mathbf{r}) \right\rangle_{vs} &= \exp(i\mathbf{k}_{\perp i} \cdot \mathbf{r}) \\ &\quad \left\{ A_j^p(\mathbf{k}_{\perp i}) \mathbf{p}^+ \exp\left[i q_j^p z \right] + B_j^p(\mathbf{k}_{\perp i}) \mathbf{p}^- \exp\left[-i q_j^p z \right] \right\} \qquad j = 1, 2, \ldots, N \end{aligned} \tag{27}$$

$$\left\langle \mathbf{E}_0^p(\mathbf{r}) \right\rangle_{vs} = \exp(i\mathbf{k}_{\perp i} \cdot \mathbf{r}) \left\{ \mathbf{p}_0^- \exp\left[-i k_{0zi}^p z \right] + R^p(\mathbf{k}_{\perp i}) \mathbf{p}_0^+ \exp\left[i k_{0zi}^p z \right] \right\} \tag{28}$$

$$\left\langle \mathbf{E}_{N+1}^p(\mathbf{r}) \right\rangle_{vs} = \exp(i\mathbf{k}_{\perp i} \cdot \mathbf{r}) X^p(\mathbf{k}_{\perp i}) \mathbf{p}_{N+1}^- \exp\left[-i k_{(N+1)zi}^p z \right] \tag{29}$$

where the superscript p stands for the polarization, either horizontal or vertical. $\mathbf{p}$ is the unit vector representing polarization. q_j is the z-component of χ_j. The subscript i is used to indicate that the wave vector is in the incident direction. R and X denote respectively the mean reflection and transmission coefficient of the stack. A_j and B_j denote respectively the mean coefficients of up-going and down-going waves in the j-th layer.

Based on this we can formulate the waves averaged w.r.t. volumetric fluctuations as

$$\left\langle \mathbf{E}_j^p(\mathbf{r})\right\rangle_v = \frac{1}{4\pi^2}\int d\mathbf{k}_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{r})\left\{A_j^{pq}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})q_j^+ \,\mathrm{e}^{iq_jz} + B_j^{pq}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})q_j^- \,\mathrm{e}^{-iq_jz}\right\} \tag{30}$$

$$\left\langle \mathbf{E}_0^p(\mathbf{r})\right\rangle_v = \exp(i\mathbf{k}_\perp\cdot\mathbf{r})\exp\left[-ik_{0zi}z\right]\mathbf{p}_0^- + \frac{1}{4\pi^2}\int \exp(i\mathbf{k}_\perp\cdot\mathbf{r})R^{pq}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})q_0^+\exp\left[ik_{0z}z\right]d\mathbf{k} \tag{31}$$

and

$$\left\langle \mathbf{E}_{N+1}^p(\mathbf{r})\right\rangle_v = \frac{1}{4\pi^2}\int \exp(i\mathbf{k}_\perp\cdot\mathbf{r})X^{pq}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})q_{N+1}^-\exp\left[-ik_{(N+1)z}z\right]d\mathbf{k}_\perp \tag{32}$$

where A_j, B_j, R and X are now integral operators representing scattering from rough interfaces. The boundary conditions associated with the above equations at the j-th interface are

$$\hat{\mathbf{n}}\times\left\langle \mathbf{E}_j(\mathbf{r}_\perp,\zeta_j)\right\rangle_v = \hat{\mathbf{n}}\times\left\langle \mathbf{E}_{j+1}(\mathbf{r}_\perp,\zeta_j)\right\rangle_v \qquad j=1,2,\dots,N \tag{33}$$

and

$$\hat{\mathbf{n}}\times\nabla\times\left\langle \mathbf{E}_j(\mathbf{r}_\perp,\zeta_j)\right\rangle_v = \hat{\mathbf{n}}\times\nabla\times\left\langle \mathbf{E}_{j+1}(\mathbf{r}_\perp,\zeta_j)\right\rangle_v \qquad j=1,2,\dots,N \tag{34}$$

The above system may be solved either numerically or by any one of the asymptotic methods available in rough surface scattering theory [5,4,41] to evaluate the mean coefficients that appear in (27)-(29).

We proceed now to the analysis of the second moments, by starting with (12). For convenience we write it in symbolic form as

$$\mathbf{E}_j = \breve{\mathbf{E}}_j + \sum_{k=1}^{N} \breve{\overline{\mathbf{G}}}_{jk} \overline{\mathbf{T}}_k^l \mathbf{E}_k^l \tag{35}$$

We take the tensor product of this equation with its complex conjugate and average w.r.t. volumetric fluctuations and obtain

$$\left\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \right\rangle = \left\langle \mathbf{E}_j \right\rangle_v \otimes \left\langle \mathbf{E}_j^* \right\rangle_v + \sum_{k=1}^{N} \sum_{k'=1}^{N} \sum_{l=1}^{N_k} \sum_{l'=1}^{N_{k'}} \left\langle \overline{\mathbf{G}}_{jk} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{jk'}^* \right\rangle_v \mathbf{K}_{kk'll'} \left\langle \mathbf{E}_k^l \otimes \mathbf{E}_k^{l'*} \right\rangle_v \tag{36}$$

where $\overline{\mathbf{K}}$ is the intensity operator of the volumetric fluctuations. Employing the weak fluctuation approximation we approximate $\overline{\mathbf{K}}$ by its leading term

$$\mathbf{K}_{kk'll'} \approx \left\langle \overline{\mathbf{T}}_k^l \otimes \overline{\mathbf{T}}_k^{l'*} \right\rangle \delta_{kk'} \delta_{ll'} \overline{\overline{\mathbf{I}}} \tag{37}$$

On substituting this in (36) we get,

$$\left\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \right\rangle_v = \left\langle \mathbf{E}_j \right\rangle_v \otimes \left\langle \mathbf{E}_j^* \right\rangle_v + \sum_{k=1}^{N} \rho_k \left\langle \overline{\mathbf{G}}_{jk} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{jk}^* \right\rangle_v \left\langle \overline{\mathbf{T}}_k \otimes \overline{\mathbf{T}}_k^* \right\rangle \left\langle \mathbf{E}_k \otimes \mathbf{E}_k^* \right\rangle_v \tag{38}$$

Next we average (38) w.r.t. the surface fluctuations to get

$$\left\langle \mathbf{E}_j \otimes \mathbf{E}_j^* \right\rangle_{vs} = \left\langle \left\langle \mathbf{E}_j \right\rangle_v \otimes \left\langle \mathbf{E}_j^* \right\rangle_v \right\rangle_s + \sum_{k=1}^{N} \rho_k \left\langle \left\langle \overline{\mathbf{G}}_{jk} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{jk}^* \right\rangle_v \right\rangle_s \left\langle \overline{\mathbf{T}}_k \otimes \overline{\mathbf{T}}_k^* \right\rangle \left\langle \mathbf{E}_k \otimes \mathbf{E}_k^* \right\rangle_{vs} \tag{39}$$

where we have used the following approximation

$$\left\langle \left\langle \overline{\mathbf{G}}_{jk} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{jk}^* \right\rangle_v \left\langle \overline{\mathbf{T}}_k \otimes \overline{\mathbf{T}}_k^* \right\rangle \left\langle \mathbf{E}_k \otimes \mathbf{E}_k^* \right\rangle_v \right\rangle_s \approx \left\langle \left\langle \overline{\mathbf{G}}_{jk} \right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{jk}^* \right\rangle_v \right\rangle_s \left\langle \overline{\mathbf{T}}_k \otimes \overline{\mathbf{T}}_k^* \right\rangle \left\langle \mathbf{E}_k \otimes \mathbf{E}_k^* \right\rangle_{vs} \tag{40}$$

We call this the 'weak surface correlation' approximation, which we will see later to be an important condition embedded in the RT approach to our problem.

As it stands this equation is very difficult to solve either analytically or numerically. Besides, one important goal for us is to investigate the conditions needed for employing the radiative transfer approach for our problem. With these in mind we introduce Wigner transforms. Note that (39) is an equation for the coherence function which is a 'space-space' quantity. On the other hand the RT equation, as we saw earlier, is an equation for the specific intensity which is a 'phase-space' quantity. Wigner transforms serve as a bridge to link these two quantities [43,9,20,30].

We introduce Wigner transforms of waves and Green's functions as

$$\varepsilon_m\left(\frac{\mathbf{r}+\mathbf{r}'}{2},\mathbf{k}\right)=\int \left\langle \mathbf{E}_m(\mathbf{r})\right\rangle_v \otimes \left\langle \mathbf{E}_m^*(\mathbf{r}')\right\rangle_{vs} \exp\left[-ik\cdot(\mathbf{r}-\mathbf{r}')\right]d(\mathbf{r}-\mathbf{r}') \tag{41a}$$

$$\varepsilon_m^s\left(\frac{\mathbf{r}+\mathbf{r}'}{2},\mathbf{k}\right)=\int \left\langle\left\langle \mathbf{E}_m(\mathbf{r})\right\rangle_v \otimes \left\langle \mathbf{E}_m^*(\mathbf{r}')\right\rangle\right\rangle_s \exp\left[-ik\cdot(\mathbf{r}-\mathbf{r}')\right]d(\mathbf{r}-\mathbf{r}') \tag{41b}$$

$$\mathcal{G}_{mn}\left(\frac{\mathbf{r}+\mathbf{r}'}{2},\mathbf{k}\,|\,\frac{\mathbf{r}_1+\mathbf{r}_1'}{2},1\right)=\int d(\mathbf{r}-\mathbf{r}')\int d(\mathbf{r}_1-\mathbf{r}_1')\,\mathrm{e}^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,\mathrm{e}^{i\mathbf{l}\cdot(\mathbf{r}_1-\mathbf{r}_1')}\left\langle\left\langle \overline{\mathbf{G}}_{mn}(\mathbf{r},\mathbf{r}_1)\right\rangle_v \otimes \left\langle \overline{\mathbf{G}}_{mn}^*(\mathbf{r}',\mathbf{r}_1')\right\rangle_v\right\rangle_s \tag{42}$$

In terms of these transforms (39) becomes

$$\mathcal{E}_m(\mathbf{r},\mathbf{k})=\mathcal{E}_m^s(\mathbf{r},\mathbf{k})+\frac{4}{(2\pi)^4}\sum_{n=1}^{N}\rho_n\int_{\Omega_n}d\mathbf{r}'\int d\boldsymbol{\alpha}\int d\boldsymbol{\beta}\,\mathcal{G}_{mn}(\mathbf{r},\mathbf{k}\,|\,\mathbf{r}',\boldsymbol{\alpha})\mathsf{T}_n(\mathbf{r}',\boldsymbol{\alpha}\,|\,\mathbf{r}'',\beta)\mathcal{E}_n(\mathbf{r}'',\beta) \tag{43}$$

with

$$\mathsf{T}_n(\mathbf{R}_1,\boldsymbol{\alpha}\,|\,\mathbf{R}_2,\boldsymbol{\beta})=\int d\mathbf{r}_1\int d\mathbf{r}_2\exp\left\{-i\boldsymbol{\alpha}\cdot\mathbf{r}_1+i\boldsymbol{\beta}\cdot\mathbf{r}_2\right\}\left\langle\overline{\mathbf{T}}\left(\mathbf{R}_1+\tfrac{\mathbf{r}_1}{2},\mathbf{R}_2+\tfrac{\mathbf{r}_2}{2}\right)\otimes\overline{\mathbf{T}}^*\left(\mathbf{R}_1-\tfrac{\mathbf{r}_1}{2},\mathbf{R}_2-\tfrac{\mathbf{r}_2}{2}\right)\right\rangle \tag{44}$$

where $\overline{\mathbf{T}}$ is the element transition operator in *n*-th layer.

The fact that our problem has translational invariance in azimuth implies the following:

$$\mathcal{E}_m(\mathbf{r},\mathbf{k})=\mathcal{E}_m(z,\mathbf{k}) \tag{45a}$$

$$\mathcal{G}_{mn}(\mathbf{r},\mathbf{k}\,|\,\mathbf{r}',\mathbf{l})=\mathcal{G}_{mn}(z,\mathbf{k}\,|\,z',\mathbf{l};\mathbf{r}_\perp-\mathbf{r}_\perp') \tag{45b}$$

Using these relations in (43) we have

$$\mathscr{E}_m\left(z,\mathbf{k}\right)=\mathscr{E}_m^s\left(z,\mathbf{k}\right)+\frac{4}{(2\pi)^4}\sum_{n=1}^{N}\rho_n\int_{z_n}^{z_{n-1}}dz'\int d\boldsymbol{\alpha}\int d\boldsymbol{\beta}\mathscr{G}_{mn}\left(z,\mathbf{k}\mid z',\boldsymbol{\alpha};0\right)\mathscr{F}_n(\boldsymbol{\alpha},\beta)\mathscr{E}_n(z',\beta) \tag{46}$$

where

$$\mathscr{G}_{mn}\left(z,\mathbf{k}\mid z',\boldsymbol{\alpha};0\right)=\int\mathscr{G}_{mn}\left(z,\mathbf{k}\mid z',\boldsymbol{\alpha};\mathbf{r}_\perp-\mathbf{r}'_\perp\right)d(\mathbf{r}_\perp-\mathbf{r}'_\perp) \tag{47}$$

$$\mathscr{F}_n(\boldsymbol{\alpha},\beta)=\overline{\mathbf{f}}(\boldsymbol{\alpha},\beta)\otimes\overline{\mathbf{f}}^*(\boldsymbol{\alpha},\beta) \tag{48}$$

$\overline{\mathbf{f}}$ is the element scattering matrix in the *n*-th layer. Since the medium is assumed to be sparse inter-particle scattering takes place in the far-field zone of each other. It is based on this fact that we have transitioned from T_n to $\mathscr{F}_n$. Also we have employed the on-shell approximation to T_n.

To proceed further we need to evaluate $\mathscr{G}_{mn}$. Note that we need to relate this system with that of RT, which involves the boundary conditions at the interfaces. Therefore we need to identify the coherence functions corresponding to up- and down-going wave functions. To facilitate this we decompose $\left\langle G_{mn}\right\rangle_v$ into its components,

$$\left\langle\overline{\mathbf{G}}_{mn}\right\rangle_v=\delta_{mn}\overline{\mathbf{G}}_m^0+\overline{\mathbf{G}}_{mn}^{uu}+\overline{\mathbf{G}}_{mn}^{ud}+\overline{\mathbf{G}}_{mn}^{du}+\overline{\mathbf{G}}_{mn}^{dd} \tag{49}$$

where the first term is the singular part of the Green's function. The superscripts *u* and *d* indicate up- and down-going elements of the waves. The other components are due to reflections from boundaries. These are formally constructed using the concept of surface scattering operators as follows [33],

$$\left\langle\overline{\mathbf{G}}_{mn}^{ab}(\mathbf{r},\mathbf{r}')\right\rangle_v^{\mu\nu}=\frac{1}{(2\pi)^4}$$
$$\int d\mathbf{k}_\perp\int d\mathbf{k}'_\perp\left\{S_{mn}^{ab}(\mathbf{k}_\perp,\mathbf{k}'_\perp)\right\}^{\mu\nu}\exp\left\{i\mathbf{k}_\perp\cdot\mathbf{r}+iaq_m^\mu(\mathbf{k}_\perp)z\right\}\exp\left\{-i\mathbf{k}'_\perp\cdot\mathbf{r}'-ibq_n^\nu(\mathbf{k}'_\perp)z'\right\} \tag{50}$$

where $\overline{\mathbf{S}}_{mn}^{ab}$ is the surface scattering operator. The superscripts *a* and *b* on *S* are used to indicate whether the waves are up-going or down-going. In the exponents, *a, b*=1 if the waves are up-going. We let *a, b* = -1 if the waves are down-going. The *z*-component of the mean propagation constants in the *n*-th layer is denoted as q_n. We recall that $\mathscr{F}_{mn}$ is the Wigner transform of $\left\langle\left\langle\overline{\mathbf{G}}_{mn}\right\rangle_v\otimes\left\langle\overline{\mathbf{G}}_{mn}^*\right\rangle_v\right\rangle_s$. The superscripts μ,ν stand for polarization, either *h* or *v*. When we use (34) to perform the Wigner transform we ignore all cross terms. In other words, we make the following approximation,

$$\mathcal{G}_{mn} = \delta_{mn}\mathcal{G}_m^0 + \mathcal{G}_{mn}^{uu} + \mathcal{G}_{mn}^{ud} + \mathcal{G}_{mn}^{du} + \mathcal{G}_{mn}^{dd} \tag{51}$$

where $\mathcal{G}_{mn}^{ab}$ is the Wigner transform of $\left\langle \left\langle \bar{\mathbf{G}}_{mn}^{ab} \right\rangle_v \otimes \left\langle \bar{\mathbf{G}}_{mn}^{ab*} \right\rangle_v \right\rangle_s$.

With the introduction of this representation for $\mathcal{G}_{mn}$ in (43) we can trace up- and down-going waves to obtain the following equations for the coherence function:

$$\begin{aligned}\mathcal{E}_m^u(z,\mathbf{k}) = \mathcal{E}_m^{su}(z,\mathbf{k}) & \\ &+\frac{4}{(2\pi)^4}\rho_m \int_{z_m}^{z} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta} \mathcal{G}_m^{>}(z,\mathbf{k} \mid z',\boldsymbol{\alpha};0) \Upsilon_m^{ua}(\boldsymbol{\alpha},\beta) \mathcal{E}_m^a(z',\beta) \\ &+\frac{4}{(2\pi)^4}\sum_{n=1}^{N} \rho_n \int_{z_n}^{z_{n-1}} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta} \mathcal{G}_{mn}^{ua}(z,\mathbf{k} \mid z',\boldsymbol{\alpha};0) \Upsilon_m^{ab}(\boldsymbol{\alpha},\beta) \mathcal{E}_n^b(z',\beta)\end{aligned} \tag{52a}$$

$$\begin{aligned}\mathcal{E}_m^d(z,\mathbf{k}) = \mathcal{E}_m^{sd}(z,\mathbf{k}) & \\ &+\frac{4}{(2\pi)^4}\rho_m \int_{z}^{z_{m-1}} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta} \mathcal{G}_m^{<}(z,\mathbf{k} \mid z',\boldsymbol{\alpha};0) \Upsilon_m^{da}(\boldsymbol{\alpha},\beta) \mathcal{E}_m^a(z',\beta) \\ &+\frac{4}{(2\pi)^4}\sum_{n=1}^{N} \rho_n \int_{z_n}^{z_{n-1}} dz' \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta} \mathcal{G}_{mn}^{da}(z,\mathbf{k} \mid z',\boldsymbol{\alpha};0) \Upsilon_n^{ab}(\boldsymbol{\alpha},\beta) \mathcal{E}_n^b(z',\beta)\end{aligned} \tag{52b}$$

Note that summation over $a,b = \{u,d\}$ is implied in the above equations. The first term in these equations, $\mathcal{E}^{sa}$, represents the contribution due exclusively to surface scattering, and has the following form:

$$\begin{aligned}\left\{\mathcal{E}_m^{sa}(z,\mathbf{k})\right\}^{\mu\nu} = 2\pi\delta\left\{k_z - \frac{1}{2}a\left[q_m^{\mu}(\mathbf{k}_\perp) + q_m^{\nu*}(\mathbf{k}_\perp)\right]\right\}\exp\left[ia\left(q_m^{\mu} - q_m^{\nu*}\right)z\right] & \\ \left\langle \left\{\Sigma_m^a\right\}^{\mu\mu'}\left\{\Sigma_m^{a*}\right\}^{\nu\nu'}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})\right\rangle_s E_{\mu' i}E_{\nu' i}^*\end{aligned} \tag{53}$$

where Σ_m^a is the amplitude of the up-going wave in the *m*-th layer after volumetric averaging is performed. This means that it is a random function of surface fluctuations. When we substitute (53) and the expressions for $\mathcal{G}_{mn}$ in (52) we find that

$$\left\{\mathcal{E}_m^a(z,\mathbf{k})\right\}_{\mu\nu} = 2\pi\delta\left\{k_z - \frac{1}{2}a\left[q_m^{\mu}(\mathbf{k}_\perp) + q_m^{\nu*}(\mathbf{k}_\perp)\right]\right\}\exp\left[ia\left(q_m^{\mu} - q_m^{\nu*}\right)z\right]\left\{\mathcal{E}_m^a(z,\mathbf{k}_\perp)\right\}_{\mu\nu} \tag{54}$$

On substituting this in (52) and differentiating w.r.t. *z* we obtain the following transport equations:

$$
\begin{aligned}
&\left\{\frac{d}{dz}-i\left[q_u(\mathbf{k}_\perp)-q_\nu^*(\mathbf{k}_\perp)\right]\right\}\mathscr{E}_{\mu\nu}^u(z,\mathbf{k}_\perp)= \\
&\qquad = 4\rho_m\int d\boldsymbol{\alpha}_\perp S_\mu^>\left(\boldsymbol{\alpha}_\perp\right)\otimes S_\nu^{>*}\left(\boldsymbol{\alpha}_\perp\right)\cdot \\
&\qquad\quad \cdot\,\mathscr{F}_{\mu\nu;\mu'\nu'}^{ua}\left\{\mathbf{k}_\perp,\frac{1}{2}\left[q_\mu(\mathbf{k}_\perp)+q_\nu^*(\mathbf{k}_\perp)\right];\boldsymbol{\alpha}_\perp,\frac{1}{2}a\left[q_{\mu'}(\boldsymbol{\alpha}_\perp)+q_{\nu'}^*(\boldsymbol{\alpha}_\perp)\right]\right\}\mathscr{E}_{\mu'\nu'}^a(z,\boldsymbol{\alpha}_\perp)
\end{aligned}
\tag{55a}
$$

$$
\begin{aligned}
&\left\{-\frac{d}{dz}-i\left[q_u(\mathbf{k}_\perp)-q_\nu^*(\mathbf{k}_\perp)\right]\right\}\varepsilon_{\mu\nu}^d(z,\mathbf{k}_\perp)= \\
&\qquad = 4\rho_m\int d\boldsymbol{\alpha}_\perp S_\mu^<\left(\boldsymbol{\alpha}_\perp\right)\otimes S_\nu^{<*}\left(\boldsymbol{\alpha}_\perp\right)\cdot \\
&\qquad\quad \cdot\,\mathscr{F}_{\mu\nu;\mu'\nu'}^{ua}\left\{\mathbf{k}_\perp,-\frac{1}{2}\left[q_\mu(\mathbf{k}_\perp)+q_\nu^*(\mathbf{k}_\perp)\right];\boldsymbol{\alpha}_\perp,\frac{1}{2}a\left[q_{\mu'}(\boldsymbol{\alpha}_\perp)+q_{\nu'}^*(\boldsymbol{\alpha}_\perp)\right]\right\}\mathscr{E}_{\mu'\nu'}^a(z,\boldsymbol{\alpha}_\perp)
\end{aligned}
\tag{55b}
$$

where summation over *a* is implied. When the superscript *a* corresponds to *u* the value of *a* in the argument of $\mathscr{F}_m$ take the value +1; on the other hand when the superscript *a* corresponds to *d* the value of *a* in the argument of $\mathscr{F}_m$ take the value -1. Since all quantities in (55) correspond to the same layer *m* we have dropped the subscript *m* in $\mathscr{F}$ and $\mathscr{E}$ to avoid cumbersome notations. To obtain appropriate boundary conditions we have to go back to the integral equation representations for $\mathscr{E}_{\mu\nu}^u$ and $\mathscr{E}_{\mu\nu}^d$, examine their behavior at the interfaces, and seek a relation between them. After considerable effort we managed to arrive at the following boundary conditions. At the *(m*-1*)*-th interface we have

$$
\mathscr{E}_m^d(z_{m-1},\mathbf{k}_\perp)=\int\left\langle\dddot{\Re}_{m-1,m}(\mathbf{k}_\perp,\mathbf{k}'_\perp)\right\rangle\mathscr{E}_m^u(z_{m-1},\mathbf{k}'_\perp)d\mathbf{k}'_\perp \tag{56}
$$

with $\dddot{\Re}=\dddot{\mathbf{R}}\otimes\dddot{\mathbf{R}}^*$ where $\dddot{\mathbf{R}}_{m-1,m}$ is the stack reflection matrix (not the local reflection matrix) for a wave incident from below on the $(m-1)$-th interface. Similarly

$$
\mathscr{E}_m^u(z_{m-1},\mathbf{k}_\perp)=\int\left\langle\dddot{\Re}_{m+1,m}(\mathbf{k}_\perp,\mathbf{k}'_\perp)\right\rangle\mathscr{E}_m^d(z_{m-1},\mathbf{k}'_\perp)d\mathbf{k}'_\perp \tag{57}
$$

where $\dddot{\Re}_{m+1,m}$ is the tensor product of stack reflection matrix for a wave incident from above on the $(m-1)$-th interface. We were able to obtain the boundary conditions only after imposing certain approximations such as the one given below. Consider the following identity:

$$\bar{\mathbf{S}}_{mm}^{du} = \bar{\mathbf{D}}_m \dddot{\bar{\mathbf{R}}}_{m-1,m} \left\{ \bar{\mathbf{S}}_m^{>} + \bar{\mathbf{S}}_{mm}^{uu} \right\} \bar{\mathbf{D}}_m \tag{58}$$

where $\bar{\mathbf{D}}_m = \mathrm{diag}\left\{ \exp(iq_h d_m), \exp(iq_v d_m) \right\}$. Notice that this is an operator relation where all elements are operators. Taking the tensor product of (58) with its complex conjugate we have

$$\begin{aligned} \bar{\mathbf{S}}_{mm}^{du} \otimes \bar{\mathbf{S}}_{mm}^{du*} = (\bar{\mathbf{D}}_m \otimes \bar{\mathbf{D}}_m^*) & \left(\dddot{\bar{\mathbf{R}}}_{m-1,m} \otimes \dddot{\bar{\mathbf{R}}}_{m-1,m}^* \right) \\ & \left[\left\{ \bar{\mathbf{S}}_m^{>} + \bar{\mathbf{S}}_{mm}^{uu} \right\} \otimes \left\{ \bar{\mathbf{S}}_m^{>} + \bar{\mathbf{S}}_{mm}^{uu} \right\}^* \right] \left(\bar{\mathbf{D}}_m \otimes \bar{\mathbf{D}}_m^* \right) \end{aligned} \tag{59}$$

Next we average (59) w.r.t. surface fluctuations and get

$$\begin{aligned} \left\langle \bar{\mathbf{S}}_{mm}^{du} \otimes \bar{\mathbf{S}}_{mm}^{du*} \right\rangle \approx (\bar{\mathbf{D}}_m \otimes \bar{\mathbf{D}}_m^*) & \left\langle \dddot{\bar{\mathbf{R}}}_{m-1,m} \otimes \dddot{\bar{\mathbf{R}}}_{m-1,m}^* \right\rangle \\ & \left\langle \left\{ \bar{\mathbf{S}}_m^{>} + \bar{\mathbf{S}}_{mm}^{uu} \right\} \otimes \left\{ \bar{\mathbf{S}}_m^{>} + \bar{\mathbf{S}}_{mm}^{uu} \right\}^* \right\rangle \left(\bar{\mathbf{D}}_m \otimes \bar{\mathbf{D}}_m^* \right) \end{aligned} \tag{60}$$

where we have approximated that the two tensor products in the middle are weakly correlated. A further approximation that we impose is given as follows:

$$\left\langle \left\{ \bar{\mathbf{S}}_m^{>} + \bar{\mathbf{S}}_{mm}^{uu} \right\} \otimes \left\{ \bar{\mathbf{S}}_m^{>} + \bar{\mathbf{S}}_{mm}^{uu} \right\}^* \right\rangle \approx \bar{\mathbf{S}}_m^{>} \otimes \bar{\mathbf{S}}_m^{>*} + \left\langle \bar{\mathbf{S}}_{mm}^{uu} \otimes \bar{\mathbf{S}}_{mm}^{uu*} \right\rangle \tag{61}$$

These are the kinds of approximations required to arrive at our boundary conditions.

3.4 Transition to Radiative Transfer

Next we have to transition from this transport equation (55) to the phenomenological radiative transfer equation discussed earlier. To accomplish this we have to link the key quantities of waves and radiative transfer, viz., coherence function and specific intensity. The relation between them is obtained by computing the Poynting vector using the two concepts. One of the fundamental results of electromagnetic wave theory is the Poynting vector given as

$$\left\langle \mathbf{S}(\mathbf{r}) \right\rangle_t = \tfrac{1}{2}\mathrm{Re}\left\{ \left\langle \mathbf{E} \times \mathbf{H}^* \right\rangle \right\}$$

For time-harmonic tranverse electromagnetic waves this becomes

$$\left\langle \mathbf{S}(\mathbf{r}) \right\rangle_t = \tfrac{\hat{\mathbf{k}}}{2\eta}\left\{ \left\langle \left| E_h(\mathbf{r}) \right|^2 \right\rangle + \left\langle \left| E_v(\mathbf{r}) \right|^2 \right\rangle \right\}$$

From Wigner transform relations we have

$$\left\langle E_\mu(\mathbf{r}) E_v(\mathbf{r}) \right\rangle = \tfrac{1}{(2\pi)^2} \int \mathcal{E}_{\mu v}(\mathbf{r}, \mathbf{k}_\perp) d\mathbf{k}_\perp \tag{62}$$

The average Poynting vector is also related to the specific intensity as

$$\left\langle \mathbf{S}(\mathbf{r}) \right\rangle = \int \hat{k}\, \mathbf{I}(\mathbf{r}, \mathbf{k}) d\Omega \tag{63}$$

The above two relations suggest that the following definition for the specific intensities

$$I_{\mu v}(z, \hat{s}) = \frac{1}{2\eta} \frac{k'^2}{(2\pi)^2} \cos\theta \mathcal{E}_{\mu v}(z, \mathbf{k}_\perp) \tag{64}$$

Now we can transition to the phenomenological RT equations. Using the relation between $\mathcal{E}$ and I we change the integration variable to solid angle and arrive at the following equations,

$$\left\{ \cos\theta \frac{d}{dz} + \eta_{ij} \right\} I_j^u(z, \hat{s}) = \int \left\{ P_{ij}^{uu}(\Omega, \Omega') I_j^u(z, \hat{s}') + P_{ij}^{ud}(\Omega, \Omega') I_j^d(z, \hat{s}') \right\} d\Omega' \tag{65a}$$

$$\left\{ -\cos\theta \frac{d}{dz} + \eta_{ij} \right\} I_j^d(z, \hat{s}) = \int \left\{ P_{ij}^{du}(\Omega, \Omega') I_j^u(z, \hat{s}') + P_{ij}^{dd}(\Omega, \Omega') I_j^d(z, \hat{s}') \right\} d\Omega' \tag{65b}$$

where $\bar{\bar{\eta}}$ is the extinction matrix and $\bar{\bar{\mathbf{P}}}$ is the phase matrix. Implicit summation over subscript *j* is assumed in (65). To facilitate comparison with the results of Ulaby et al [38] and Lam and Ishimaru [14] we have used a modified version of Stokes vector [11]. Instead of the standard form $\{I,Q,U,V\}$ we use $\{(I+Q)/2,(I-Q)/2,U,V\}$. The subscript of I in (65) denotes the element number of our modified Stokes vector. Although the structure of this equation is identical to that of the RT (equation (2)), the elements of the phase matrix and the extinction matrices are not the same. The primary reason is because of the differences in the real part of the mean propagation constants of horizontally and vertically polarized waves. On assuming that $q_h' = q_v' = k_{mz}'$ we obtain the following expressions for the extinction and phase matrices:

$$\overline{\overline{\eta}} = -\cos\theta \ \mathrm{diag}\{2q_v'', 2q_h'', q_v''+q_h'', q_v''+q_h''\} \tag{66}$$

$$P_{ij}^{ab}(\Omega,\Omega') = \mathscr{P}_{ij}\{\mathbf{k}_\perp, ak\cos\theta; \mathbf{k}'_\perp, bk\cos\theta'\} \tag{67}$$

where

$$\begin{array}{llll}
\mathscr{P}_{11} = \left\langle |f_{vv}|^2 \right\rangle & \mathscr{P}_{12} = \left\langle |f_{vh}|^2 \right\rangle & \mathscr{P}_{13} = \left\langle \mathrm{Re}(f_{vv} f_{vh}^*) \right\rangle & \mathscr{P}_{14} = -\left\langle \mathrm{Im}(f_{vv} f_{vh}^*) \right\rangle \\
\mathscr{P}_{21} = \left\langle |f_{hv}|^2 \right\rangle & \mathscr{P}_{22} = \left\langle |f_{hh}|^2 \right\rangle & \mathscr{P}_{23} = \left\langle \mathrm{Re}(f_{hv} f_{hh}^*) \right\rangle & \mathscr{P}_{24} = -\left\langle \mathrm{Im}(f_{hv} f_{hh}^*) \right\rangle \\
\mathscr{P}_{31} = 2\left\langle \mathrm{Re}(f_{vv} f_{hv}^*) \right\rangle & \mathscr{P}_{32} = 2\left\langle \mathrm{Re}(f_{vh} f_{hh}^*) \right\rangle & \mathscr{P}_{33} = 2\left\langle \mathrm{Re}(f_{vv} f_{hh}^*) \right\rangle & \mathscr{P}_{34} = -\left\langle \mathrm{Im}(f_{vv} f_{hh}^*) \right\rangle \\
 & & & \qquad +\left\langle \mathrm{Re}(f_{vh} f_{hv}^*) \right\rangle \\
\mathscr{P}_{41} = 2\left\langle \mathrm{Im}(f_{vv} f_{hv}^*) \right\rangle & \mathscr{P}_{42} = 2\left\langle \mathrm{Im}(f_{vh} f_{hh}^*) \right\rangle & \mathscr{P}_{43} = \left\langle \mathrm{Im}(f_{vv} f_{hh}^*) \right\rangle & \mathscr{P}_{44} = \left\langle \mathrm{Re}(f_{vv} f_{hh}^*) \right\rangle \\
 & & \qquad +\left\langle \mathrm{Im}(f_{vh} f_{hv}^*) \right\rangle & \qquad -\left\langle \mathrm{Re}(f_{vh} f_{hv}^*) \right\rangle
\end{array} \tag{68}$$

We have suppressed the arguments for brevity. For instance,

$$\mathscr{P}_{13}\{\mathbf{k}_\perp, ak\cos\theta; \mathbf{k}'_\perp, bk\cos\theta'\}$$

$$= \left\langle \mathrm{Re}\left\{ f_{vv}\left(\mathbf{k}_\perp, ak\cos\theta; \mathbf{k}'_\perp, bk\cos\theta'\right) f_{vh}^*\left(\mathbf{k}_\perp, ak\cos\theta; \mathbf{k}'_\perp, bk\cos\theta'\right)\right\}\right\rangle \tag{69}$$

f 's are the elements of the scattering matrix $\overline{\mathbf{f}}$ of particles defined as follows:

$$\mathbf{E}_s = \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{r}\overline{\mathbf{f}}\,\mathbf{E}_i$$

In the $\{h,v\}$ basis $\overline{\mathbf{f}}$ is given as

$$\overline{\mathbf{f}} = \begin{bmatrix} f_{vv} & f_{vh} \\ f_{hv} & f_{hh} \end{bmatrix}$$

Note that these transport equations (65) are identical to those of classical RT equations (2) that we described in Section 2. Thanks to our statistical wave approach we now have explicit expressions for the extinction matrix and phase matrix in terms of the statistical parameters of the problem. Let us now next turn our attention to the boundary conditions (BC). In our wave approach we obtained BCs in terms of 'stack' reflection matrix $\dddot{\overline{\mathbf{R}}}$, whereas in the RT approach the BCs are given in terms of the local interface reflection matrices. We can readily reconcile with this apparent difference. Note that the BC in the wave approach forms a closed system whereas in the RT approach it is open (linked to adjacent layer intensities). Let us take a look at the BC at the (m-1)-th interface. $\dddot{\overline{\mathbf{R}}}_{m-2,m}$ can be expressed in terms of $\overline{\mathbf{R}}_{m-2,m-1}$ as follows,

$$\dddot{\overline{\mathbf{R}}}_{m-1,m} = \overline{\mathbf{R}}_{m-1,m} + \overline{\mathbf{T}}_{m,m-1}\left[\overline{\mathbf{I}} - \dddot{\overline{\mathbf{R}}}_{m-2,m-1}\overline{\mathbf{D}}_{m-1}\overline{\mathbf{R}}_{m,m-1}\right]^{-1}\dddot{\overline{\mathbf{R}}}_{m-2,m-1}\overline{\mathbf{D}}_{m-1}\overline{\mathbf{T}}_{m-1,m} \tag{70}$$

This is the relation between the stack reflection coefficients of adjacent interfaces. The $\overline{\mathbf{R}}$ and $\overline{\mathbf{T}}$ are local (single interface) reflection and transmission matrices at the $(m-1)$-th interface. On operating $\mathbf{E}_m^u$ with (70) we get

$$\mathbf{E}_m^d = \overline{\mathbf{R}}_{m-1,m}\mathbf{E}_m^u + \overline{\mathbf{T}}_{m,m-1}\mathbf{E}_{m-1}^d \tag{71}$$

Notice that this boundary condition now involves only local interface Fresnel coefficients. Take the tensor product of (71) with its complex conjugate and average w.r.t. surface fluctuations. Employing the Wigner transform operator on this, we obtain a boundary condition at the (m-1)-th interface similar to that of the RT system. However, the reflection and transmission matrices used in the RT system correspond to unperturbed medium as opposed to the average medium as in the case of the wave approach.

Similarly we write $\dddot{\overline{\mathbf{R}}}_{m+1,m}$ in terms of $\dddot{\overline{\mathbf{R}}}_{m+2,m+1}$ and hence obtain the BC at the m-th interface as

$$\mathbf{E}_m^u = \overline{\mathbf{R}}_{m+1,m}\mathbf{E}_m^d + \overline{\mathbf{T}}_{m,m+1}\mathbf{E}_{m+1}^u \tag{72}$$

Take the tensor product of (72) with its complex conjugate and average w.r.t. surface fluctuations. Employing the Wigner transform operator on this, we obtain boundary condition at the m-th interface identical to (3) (after making the approximation as before).

4. Results and Discussion

The main goal of this report is to critically examine the radiative transfer (RT) approach to remote sensing of layered random media with rough interfaces. Such problems are often encountered in applications. Several assumptions are embedded in the radiative transfer approach. There are numerous works in the literature on the study of radiative transfer theory and the underlying assumptions. However, all have dealt with unbounded geometry or bounded geometry with nonscattering boundaries. Our interest in this report is on the problem of layered random media with irregular scattering boundaries. There does not exist any critical study of the RT approach to this kind of geometry. Our study has shown that there are additional conditions embedded in the RT approach to this problem than for the problem with unbounded or nonscattering geometries. These facts are not well known to the users of the RT approach. One purpose of this report is to inform the remote sensing community about these additional conditions so that they have a good idea on when the RT approach is an acceptable model for the application at hand.

To enable this study we developed a statistical wave theory for the combined problem of layered random media with rough interfaces. Such a foundation is essential for this study because there is no suitable wave theory in the literature that is suitable for multiple scattering for our problem. From this study we find that the coherent waves for our problem behave like waves in a layered homogeneous media with planar interfaces. The propagation constants of this layered media are primarily determined by the statistical properties of the local medium. It is weakly dependent on interface roughness. Its dependence on medium fluctuations of other layers is of higher order. In contrast, the effective reflection coefficients are influenced by not only statistical properties of the local interface but also on the statistical properties of the adjacent media. Its dependence on the surface roughness of other interfaces and media are of higher order. We notice that diffuse scattering is fairly complicated because of volume-surface interactions. In recent years there has been a proposition that the combined problem of volumetric scattering and surface scattering be split into two parts: one due to volumetric scattering and the other surface scattering. The surface scattering part is due to the layered structure with rough interfaces and homogeneus media with effective permittivities. The volumetric scattering is from the layered random medium with unperturbed at interfaces. It is clear from our results that this kind of splitting is not, in general, possible unless the fluctuations of the problem are very weak.

The results obtained in this report apply to two situations. Below we consider them separately:
<u>(a) Time-Varying Problem:</u> This means that the parameters of the problem vary with time. However, an important assumption must be made to simplify the analysis. The time constant associated with the parameter fluctuations of the problem should be much larger

than the time constant associated with signal of the incident wave. The time average is taken over a period much larger than the time constant associated with the medium fluctuations (that includes volumetric and surface fluctuations). Now we impose the ergodicity hypothesis and equate time-averages to ensemble-averages. One such application where this situation occurs is in electromagnetic wave propagation and scattering in atmosphere.
(b) Time Independent Problem: Here all the parameters are independent of time. However they undergo spatial variations. The statistical aspects of the problem enter through spatial fluctuations. Therefore all averages are ensemble averages. Obviously then the question of ergodicity does not arise here.

The transport equations as derived in (55) along with the boundary conditions (56) and (57) are important results of the report. This system describes the behaviour of up and downward travelling coherence functions in each layer of the problem. Several important physical quantities can be directly calculated using these coherence functions.
(a) Poynting-Stokes Tensor: The Poynting-Stokes tensor [21,22], a key quantity in radiative transfer, is related to our coherence function as follows:

$$\overline{\mathbf{P}}(z) = \frac{1}{2\eta(z)}\frac{1}{(2\pi)^2}\left\{ \int_{\Omega^+} \hat{k}^+ \times \mathcal{E}^u(z,\mathbf{k}_\perp)d\mathbf{k}_\perp + \int_{\Omega^-} \hat{k}^- \times \mathcal{E}^d(z,\mathbf{k}_\perp)d\mathbf{k}_\perp \right\} \tag{73}$$

where $\eta(z)$ is the intrinsic impedance of the medium where z is located. $\hat{k}^+$ and $\hat{k}^-$ are the unit propagation vectors associated with $\mathcal{E}^u$ and $\mathcal{E}^d$ respectively. Ω^+ and Ω^- indicate that the domain of integration is the upper and lower hemisphere respectively.
(b) Specific Intensity: We showed the relationship between coherence function and the specific intensity governed by the phenomenological radiative transfer equation (65). Details are given below:

$$\begin{bmatrix} I_{vv} \\ I_{hh} \\ U \\ V \end{bmatrix} = \frac{1}{2\eta}\frac{k^2}{(2\pi)^2}\cos\theta \begin{bmatrix} \mathcal{E}_{vv} \\ \mathcal{E}_{hh} \\ 2\mathrm{Re}\left(\mathcal{E}_{vv}\mathcal{E}_{hh}^*\right) \\ 2\mathrm{Im}\left(\mathcal{E}_{vv}\mathcal{E}_{hh}^*\right) \end{bmatrix} \tag{74}$$

(c) Average Poynting Vector: This quantity which is vital for radiation budget computation [36,18] is derived from the coherence function as follows:

$$\langle \mathbf{S}(z) \rangle = \frac{1}{2\eta(z)}\frac{1}{(2\pi)^2}\sum_{\mu}\left\{ \int_{\Omega^+} \hat{k}^+ \mathcal{E}^u_{\mu\mu}(z,\mathbf{k}_\perp)d\mathbf{k}_\perp + \int_{\Omega^-} \hat{k}^- \mathcal{E}^d_{\mu\mu}(z,\mathbf{k}_\perp)d\mathbf{k}_\perp \right\} \tag{75}$$

(d) Bistatic Scattering Coefficient: In our problem the system is excited by a plane wave incident on the zero-th interface from above. The bistatic scattering coefficient, σ^r in Region 0 (in the direction $\mathbf{k}_\perp$) is given as

$$\sigma^r(\mathbf{k}_\perp,\mathbf{k}_{\perp i}) = \frac{\cos\theta}{\cos\theta_i}\frac{1}{(2\pi)^2}\int \left\langle \mathcal{X}_{01}(\mathbf{k}_\perp,\mathbf{k}'_\perp)\right\rangle \mathcal{E}_1^u(z,\mathbf{k}'_\perp)d\mathbf{k}'_\perp \tag{76}$$

where $\mathcal{X}_{01} = X_{01}\otimes X_{01}^*$. Suppose the incident plane wave is horizontally polarized. Then the *hh* component of LHS represents σ_{hh}^r; The *vv* component represents σ_{vh}^r. Similarly if the incident wave is vertically polarized, the *hh* component of RHS represents σ_{hv}^r; the *vv* component represents σ_{vv}^r. We have used the superscript *r* to indicate that this quantity is a reflection type scattering coefficient since the source and observation points are located in the Same Region 0. This kind of scattering coefficient is extensively used in remote sensing applications [38,8,1,37]. Next suppose that the observation point is in Region $N+1$ while the source is still in Region 1. Now the bistatic scattering coefficient is given as

$$\sigma^t(\mathbf{k}_\perp,\mathbf{k}_{\perp i}) = \frac{\cos\theta}{\cos\theta_i}\frac{1}{(2\pi)^2}\int \left\langle \mathcal{X}_{N+1,N}(\mathbf{k}_\perp,\mathbf{k}'_\perp)\right\rangle \mathcal{E}_N^d(z_N,\mathbf{k}'_\perp)d\mathbf{k}'_\perp \tag{77}$$

We call this the transmission scattering coefficient and indicate it with the superscript *t*. This quantity is quite useful in a variety of imaging applications.

<u>(e) Passive Remote Sensing:</u> The key quantity of interest in passive remote sensing is emissivity *e* [8,38,1] given as

$$e_p = 1 - r_{pc} - r_{pi} \qquad\qquad p = \{h, v\} \tag{78}$$

where r_{pc} and r_{pi} are the coherent and incoherent reflectivities defined as shown below.

$$r_{pc} = \sum_q \left|\left\langle \dddot{R}_{00}(\mathbf{k}_\perp,\mathbf{k}_{\perp i})\right\rangle\right|^2 \tag{79a}$$

$$r_{pi} = \sum_q \int \sigma_{pq}^r(\mathbf{k}_\perp,\mathbf{k}_{\perp i})d\mathbf{k}_\perp \tag{79b}$$

$\dddot{R}_{00}$ is the stack reflection coefficient. σ_{pq}^r is the reflection scattering coefficient defined earlier.

Indeed there are other useful physical quantities [8,19,13] that can be obtained from the coherence function derived in (55). We have just given some examples relevant to remote sensing problems. In order to understand the foundations of the radiative transfer approach we made the transition from the governing equation for coherence function to the phenomenological radiative transfer equations. We made the transition from statistical wave theory to radiative transfer theory by employing the Wigner transform and computing the average Poynting vector by using wave theoretical methods and transport theoretical concepts.

Having made the transition from statistical wave theory to radiative transfer theory it is now instructive to itemize the assumptions. The three basic conditions required are:

I. Quasi-stationary field approximation

II. Sparse distribution of scatterers
III. The number of particles in each layer is large

These are the well-known conditions necessary for the unbounded random medium problem. However, if the medium is bounded we need to impose additional conditions. We found that the extinction coefficients calculated in the wave approach and the RT approach are different and only after further approximations can they be made to agree with each other. The additional condition required for our bounded random medium problem is:

IV. Layer thickness must be of the same order or greater than the mean free path of the layer.

When the interfaces are randomly rough we further need the following conditions.

V. Weak surface correlation approximation.
VI. All fluctuations of the problem are statistically independent of each other.

The above are the essential conditions associated with the RT approach. There are several secondary assumptions that we have employed for simplifying the analysis and discussion. They are given as follows:

1. The location and orientation are the only randomly varying aspects of the media.
2. All particles in a particular layer are of the same type and are uniformly distributed. However they can be of different type in different layers.
3. The problem is translationally invariant and isotropic in the azimuth.
4. The problem is assumed to be time-independent.
5. If the problem happens to be time-varying the following additional conditions need to be imposed: (a) The time constant of statistical parameters is much longer than that of the signal, (b) The problem is ergodic.
6. The interfaces of the random medium are parallel planes on the average.

Let us make some remarks on the secondary assumptions.

Assumption 1: One can add to it other variations such as size, material properties, etc., at the price of more complicated analysis.

Assumption 2: All particles need not be of the same type. We can have multi-species in each layer. The distributions can be arbitrary. It is not necessary for them to be uniform. However, such details will add to the complexity of the results.

Assumption 3: While translational invariance is an important assumption for our analysis, the statistical fluctuations need not be isotropic in azimuth. Layered random media without translational invariance is a very complex problem not considered in this report.

We can indeed take into account anisotropy in statistical fluctuations. However, we will have a tensor form of effective permittivity and associated mode patterns.

Assumption 4: Time-independence is another assumption intended to simplify the analysis.

Assumption 5: The results obtained in the report apply to time-varying case provided additional assumptions are imposed.

Assumption 6: This is an important assumption necessary for our analysis. However this assumption is not essential for the main character of our results.

We would like to make a couple of more remarks before ending the discussion of our assumptions: (a) In RT theory the medium is assumed to be sparse and hence the "refraction effects" of the fluctuations are ignored. Thus in the boundary conditions we should use the background medium parameters rather than the effective medium parameters as derived in our statistical wave theory. (b) To arrive at (65) we have ignored the contribution of evanescent modes.

Recently Mishchenko et al. [24] (hereafter referred to as MTL for brevity) derived the vector radiative transfer equation (VRTE) for a bounded discrete random medium using a rigorous microphysical approach. This enabled them to identify the following assumptions embedded in the VRTE.

1. Scattering medium is illuminated by a plane wave.
2. Each particle is located in the far field zone of all other particles and the observation point is also located in the far field zones of all the particles forming the scattering medium.
3. Neglect all scattering paths going through a particle two or more times (Twersky approximation).
4. Assume that the scattering system is ergodic and averaging over time can be replaced by averaging over particle positions and states.
5. Assume that (i) the position and state of each particle are statistically independent of each other and those of all other particles and (ii) spatial distribution of the particles throughout the medium is random and statistically uniform.
6 Assume that the scattering medium is convex.
7. Assume that the number of particles *N* forming the scattering medium is very large.
8. Ignore all the diagrams with crossing connections in the diagrammatic expansion of the coherency dyadic.

First, notice that the MTL dealt with the problem having nonscattering boundary. For this problem, only the first three of our essential conditions, and the first five of our secondary conditions should apply. The list of assumptions given by MTL consists of a combination of both essential and secondary conditions. Furthermore, the conditions that MTL obtained are not identical with the ones that we derived even when restricting attention to the random medium problem with nonscattering boundary. This is because the methods we employed are different. Here below we address the differences and provide explanation for the discrepancies.

MTL 2: MTL have made deliberate use of the far field approximation. We have not explicitly used this approximation. Instead we have used the more general quasi-uniform field approximation. To illustrate this consider electric fields at two points $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{E}(\mathbf{r}_1) = \mathbf{A}(\mathbf{r}_1)e^{i\mathbf{k}\cdot\mathbf{r}_1} \qquad \mathbf{E}(\mathbf{r}_2) = \mathbf{A}(\mathbf{r}_2)e^{i\mathbf{k}\cdot\mathbf{r}_2} \tag{80}$$

The coherence function of these fields is given as

$$\left\langle \mathbf{E}(\mathbf{r}_1) \otimes \mathbf{E}^*(\mathbf{r}_2) \right\rangle = \left\langle \mathbf{A}(\mathbf{r}_1) \otimes \mathbf{A}^*(\mathbf{r}_2) \right\rangle e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)} \tag{81}$$

In quasi-stationary field approximation we assume that

$$\left\langle \mathbf{A}(\mathbf{r}_1) \otimes \mathbf{A}^*(\mathbf{r}_2) \right\rangle = \mathscr{A}((\mathbf{r}_1 + \mathbf{r}_2)/2) \tag{82}$$

where $\mathscr{A} := \left\langle \mathbf{A} \otimes \mathbf{A}^* \right\rangle$. In contrast, MTL employed the far-field approximation under which A is independent of position vectors. For our problem of layered random media explicit implementation of far field approximation is unrealistic.

MTL 3: We have not explicitly used the Twersky approximation. Instead we made use of the fact that the number of particles is large and hence approximated conditional averages by unconditional averages. Incidentally, we arrive at the same result as that of MTL for the unbounded random- medium problem.

MTL 6: In our problem we have distinct scattering boundaries and the character of the waves exiting or entering them are explicitly contained in the boundary conditions. Hence convexity of the scattering medium is not a necessary condition for us.

MTL 8: In our approach we used the weak fluctuation criterion and retained only the leading term of the intensity operator. Ladder term is the leading term. Crossed terms are higher order terms which have contribution only in the backscattering direction. Hence such terms do not appear in our results.

Let us reiterate that MTL [24] with whom we made detailed comparison have considered a bounded random medium with nonscattering boundary. Ours is a significantly more complicated problem because of the layered structure and scattering interfaces with irregular geometry. This means that our problem contains volume surface interactions because of multiple scattering. Nobody has critically examined the RT approach to this problem. By following a systematic and rigorous approach we found that the RT approach to our problem requires additional conditions beyond those required for the problem with nonscattering boundaries. It is expected that some of our conditions are equivalent to those of MTL. The key results of the report lie in the extra conditions. These additional conditions that are special to our geometry are: (a) The layer thickness must be of the same order or larger than the mean free path of the medium (b) Weak surface correlation approximation. (c) All volumetric fluctuations and surface fluctuations are independent of each other. These are the essential extra conditions. Following are some secondary extra conditions that we have imposed to arrive at our results. The surface fluctuations are: (a) translationally invariant and isotropic in azimuth, (b) statistically homogeneous, and (c) single-valued. These secondary assumptions were employed in order to simplify the analysis and discussion.

The existence of these extra conditions means that RT approach, as it is popularly conceived, may be applied to only a limited type of layered random media. In other words, the accuracy which can be obtained using RT theory depends on the geometry of the problem. For certain geometries such as the one discussed in this report the results obtained using RT approach can be grossly inaccurate. We have obtained these results using a systematic analysis of the macroscopic Maxwell's equations. Nobody else has observed that additional conditions are involved in the RT approach to layer geometry than for the unbounded geometry of with nonscattering boundaries. In view of these remarks, the results of this report are important to the remote sensing community. Until now, RT theory has been taken as a fundamental law and applied to variety of random media with complex geometries. This report shows that under what conditions RT theory may be used to layered random media.

5. Conclusion

To summarize, we have enquired into the assumptions involved in adopting the radiative transfer approach to scattering from layered random media with rough interfaces. To facilitate this enquiry we adopted a wave approach to this problem and derived the governing equations for the first and second moments of the wave fields. We employed Wigner transforms and transitioned to the system corresponding to that of radiative transfer approach. In this process we found that there are more conditions implicitly involved in the RT approach to this problem than it is widely believed to be sufficient. With the recent development of fast and efficient algorithms for scattering computations and the enormous increase in computer resources it is now feasible to take an entirely numerical approach to this problem without imposing any approximations. In spite of such developments, to keep the size of the problem manageable only special cases have been studied thus far [10,31,29,33]. Hence it is very much of relevance, interest and convenience to apply the RT approach to these problems. However, one should keep in mind the assumptions involved in such an approach. Otherwise interpretations of results based on RT theory can be misleading.

6. References

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