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COMPLEX VARIABLE METHODS FOR SHAPE SENSITIVITY OF FINITE ELEMENT MODELS (PREPRINT)

Andrew Voorhees, Ronald Bagley, and Harry Millwater

University of Texas at San Antonio

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Complex Variable Methods for Shape Sensitivity of Finite Element Models

Andrew Voorhees, Ronald Bagley, Harry Millwater

Abstract

Complex variable methods have some potential advantages over classical finite differencing methods for sensitivity analysis. Two methods, complex Taylor series expansion and Fourier differentiation, are applied and compared to central differencing for shape sensitivity analysis. A two dimensional finite element model with an analytical solution is chosen for use in comparing the accuracy of the methods. It is found that for the accuracy of the model chosen, the error in the sensitivities is primarily defined by the error in the solution, not the error in the sensitivity method.

Introduction

Finite element analysis is a powerful tool in computational engineering. The method allows for the approximate solution of boundary value partial differential equations. Given enough computational power it is possible to solve highly non-linear equations, on extremely complicated and intricate domains, with arbitrary boundary conditions [1]. Finite element analysis has become an important part of the design process for mechanical engineers. Accurate numerical solutions can reduce the need for costly laboratory experiments. Furthermore, the effect that a small design change has on the performance of the model can be calculated quickly, avoiding the need to build new test specimens.

The effect that a small design change has on the output parameters of the model can be characterized through a process known as sensitivity analysis. A sensitivity is in fact identical to a partial derivative, in that it quantifies the change in one parameter due to a change in another parameter, with all other inputs held constant. This means that sensitivity analyses can be performed using simple numerical differentiation techniques. For design work, the most important type of sensitivity is the shape sensitivity [2]. A shape sensitivity tells the designer what effect a small change in the size or shape of the domain will have on the output. Examples of shape sensitivities include the sensitivity of the lift generated by an aircraft wing to the shape of the wing's cross section, or the sensitivity of the stress in a beam due to a change in it's length.

Sensitivities are easily calculated through numerical differentiation techniques. Traditionally, the numerical differentiation method of choice has been finite differencing. Finite differencing requires that a function be evaluated at additional sample points and the derivative of the function can be evaluated by calculating the difference in the function's value between two of the sample points. Central differencing (CD) is a widely used form of finite differencing chosen for it's increased accuracy. Over the last ten years, alternative numerical differentiation techniques have emerged for use in sensitivity analysis. Two of these methods are complex Taylor series expansion (CTSE), also referred to as the complex step derivative method, and Fourier differentiation. These methods offer more accurate and stable derivatives compared to CD.

CTSE was first described by Lyness and Moler in the late 1960's [3,4]. It reemerged as a tool for engineering analysis with a paper by Squire and Trapp in 1998 [5]. Since then it has been used in a wide variety of engineering fields including computational fluid dynamics, dynamic system optimization and many more [6-13]. In all of these fields CTSE has offered a great improvement in accuracy over CD. However, CTSE was only found to offer similar accuracy to CD for use in the calculation of shape sensitivity problems for one dimensional and two dimensional finite element models [13].

FD was also developed by Lyness in the late 60's and early 70's[3,4,15]. The method has was further described by Henrici and more recently Bagley [16,17]. The method utilizes additional sample points in the complex plane and an FFT routine to calculate derivatives including high order derivatives with exceptional accuracy. To date the method has not been widely used for the determination of sensitivities for engineering problems.

Although the scope of this paper does not cover them, several other sensitivity methods that do not rely on the evaluation of sample points have come into use in the engineering community [18-24]. These codes are typically more difficult to apply to problems in that they require extra coding or the derivation and solution of new equations rather than simply generating extra sample points and applying simple numerical differentiation formulas. For the most part they are restricted to problems in which high-dimensional gradients need to be calculated. The first of these methods is automatic differentiation. This method is based on the concept of the chain rule, and the fact that a large code thought of as a single function composed of several small functions each having its own partial derivative [18]. Thus by tracking the use of each small function in a code, i.e. every multiplication or subtraction operation, and storing the derivative information, sensitivities can be obtained by carefully applying the chain rule. Fortunately, several automatic differentiation codes for a variety of programming languages already exist. These include ADIFOR and ADI-C codes for FORTRAN and C respectively [25,26]. This method has been found to be more accurate than the sampling methods because each derivative evaluation can be done symbolically, which means that the error will due to machine round-off. Automatic differentiation is also very efficient for computing high-dimensional gradients. Direct differentiation and the adjoint method are two additional methods [20]. In direct differentiation, sensitivities are calculated through linearization and discretization of the original differential equations to produce discrete linearized state equations that can be solved to produce the desired sensitivities. The adjoint method is similar, except the linearization is replaced with an adjoint operation. These methods have been found to be very useful in the field of CFD where problems often involve large numbers of input variables [20].

The goal of this paper is to demonstrate the use of complex variable methods for the calculation of shape sensitivities for a simple two-dimensional finite element model. The two dimensional plane stress elastic model of a thick walled cylinder under uniform boundary pressure will be used as a numerical example. This problem has an analytical solution, which can be differentiated to determine the shape sensitivities. The sensitivities calculated by CTSE and FD will be compared to the analytical sensitivities as well as those calculated by CD.

Methodology

Numerical Differentiation

Numerical differentiation is a process through which an estimate of a function's derivative can be obtained. A derivative is defined as the limit of the change in a function's value across two different points, as the distance between the two points goes to zero.

$$f'(x_o) = \lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o} \tag{1}$$

Finite differencing methods calculate derivatives by estimating the limit in eq. 1 as a difference between a function evaluated a two distinct points located a distance *h* apart.

$$f'(x_o) \approx \frac{f(x_o + h) - f(x_o)}{h} \tag{2}$$

This distance, h, is often called the step size. When h is positive, the method is referred to as forward differencing. When h is negative it is called backwards differencing. When the forward difference and the backwards difference are averaged, the method is called central differencing. The equation for central differencing is as follows.

$$f'(x_o) \approx \frac{f(x_o + h) - f(x_o - h)}{2h} \tag{3}$$

The approximation of the derivatives as a difference between two non-identical numbers leads to error due to the truncation of terms in the functions Taylor series. This error can be eliminated by making the step size as small as possible. However, as the step size gets very small, a new source of error arises. This new error is round-off error and it is due to the fact that a computer cannot accurately calculate a small difference between two large numbers. This means that for finite differencing there is a lower limit on the step size and also a limit on the maximum achievable accuracy.

For the forward differencing method all Taylor series terms above the first order term are ignored. This means that the order of accuracy for a given step size is $O(h)$. By using CD all the even order terms in the Taylor series cancel out and the accuracy of the method becomes $O(h^2)$. The increased accuracy of CD is the reason it has become the standard method for sensitivity calculations.

Higher order derivatives can also be calculated through CD, by using additional sample points. The formula for the second derivative is.

$$f^{(2)}(x_o) \approx \frac{f(x_o + h) - 2f(x_o) + f(x_o - h)}{h^2} \tag{4}$$

Formulae exist for derivatives above second order but are not printed here. One of the problems with CD is that the calculation of higher order derivatives requires more sample points and more difference operations. Each additional difference operation results in an increase in the round-off error, which further restricts the lower limit of *h*. This means that CD is not a good choice for the calculation of higher order derivatives.

CTSE is another numerical differentiation method. CTSE uses the orthogonality of the real and imaginary axes of the complex plane to calculate derivatives with fewer difference operations and in turn less round-off error when compared to CD. CTSE is capable of calculating both the first and second order derivatives from a single sample point located at x_o+ih. The formulae for the derivatives can be derived from the Taylor series representation of the function evaluated at the complex sample point.

$$f(x_o+ih)=f(x_o)+f^{(1)}(x_o)\cdot\frac{i\cdot h}{1!}+f^{(2)}(x_o)\cdot\frac{(i\cdot h)^2}{2!}+f^{(3)}(x_o)\cdot\frac{(i\cdot h)^3}{3!}+\dots \tag{5}$$

Taking the imaginary part of both sides of eq. 5 and solving for the first derivative will result in an approximation with accuracy $O(h^2)$.

$$f'(x_o)\approx\frac{\mathrm{Im}(f(x_o+ih))}{h} \tag{6}$$

It is noted that for the first derivative no difference operation is needed. This means that the step size can be made arbitrarily small with no concern about increasing round-off error. Taking the real part of eq. 5, the formula for the second derivative with error $O(h^2)$ can be derived.

$$f'(x_o)\approx\frac{2(f(x_o)-\mathrm{Re}(f(x_o+ih)))}{h^2} \tag{7}$$

It is noted that the second derivative contains a difference operation meaning that round-off error will be a problem if *h* is set too small. By using more sample points it is possible to solve equation 5 to obtain the higher order derivatives.

FD can be derived from the Cauchy integral formula.

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \tag{8}$$

which, relates any nth order derivative to a simple contour integral in the complex plane. The form of this integral is identical to the form of a Fourier integral, and hence it can be evaluated using an FFT routine. If the function of interest is evaluated at N sample points along a circular contour in the complex plane centered on the initial point, a vector of the sampled data can be run through an FFT routine, and the output will be the first N terms in the functions Taylor series. The n^{th} order derivative of the function can then be calculated from the Taylor series coefficients by using the following relationship.

$$f^{(n)}(z_o) = \frac{a_n n!}{h^n} \tag{9}$$

where a_n is the n^{th} Taylor series coefficient. For more information on Fourier differentiation see Bagley, 2006 [17].

Finite Element Analysis

A two-dimensional finite element code capable of solving the 2-D equations of elasticity under the assumptions of plane stress was written in Matlab. The code uses second order shape functions and six-node triangular elements. The equations for the shape functions in terms of the natural coordinates can be found in Huebner, on page 543 [27]. The use of natural coordinates allows for all numerical integration to be replaced with algebraic manipulation, which greatly improves the run time of the program. Geometry and meshes were created using the Comsol finite element package and were then imported into Matlab. After importation, the edges of the elements were reconstructed so that all element edges were linear, and all non-vertex nodes were placed on the midpoint of the edges. The code used a conjugate gradient solver with a minimum error of 1e-8. After solving the global system of equations for the displacements, the stresses at each of the nodes were calculated on an element by element basis using the equations of elasticity, second order shape functions and the nodal displacements. Since, the nodal stress calculated from one element does not always agree with the nodal stress calculated using another element, stress averaging was used to smooth the stress field. The code then calculated the principal stresses and the Von Mises stresses using the stresses at the nodal points.

In order to perform the shape sensitivity calculations, a small step must be added to the nodal coordinates. This was done by first identifying every node that was located on the geometry feature that was being sampled. For instance, if the sensitivity of the solution to the radius of a hole was being calculated, the program would identify each node that was located on the surface of the hole. The code would then add a small step (complex or real depending on the method) to the nodal coordinate of each of the identified nodes. All other nodes in these elements would remain the same. If the step size chosen for the numerical differentiation technique is very large, be it real or imaginary, the elements may become distorted, which can lead to poor numerical results, and as such the step sizes and sampling radii of the differentiation methods were typically kept much smaller than the edge lengths of the elements.

Numerical Example: Thick Walled Cylinder

In order to test the accuracy of the three numerical differentiation techniques it is necessary to find a problem with an analytical solution. The thick walled cylinder under uniform boundary pressure is such a problem. The equations that govern the stress through the thickness of the cylinder are given in eq. 10 [28],

$$\sigma_r = \frac{r_1^2 r_2^2 p}{r_2^2 - r_1^2}\frac{1}{r^2} - \frac{r_2^2 p}{r_2^2 - r_1^2}$$

$$\sigma_\theta = -\frac{r_1^2 r_2^2 p}{r_2^2 - r_1^2}\frac{1}{r^2} - \frac{r_2^2 p}{r_2^2 - r_1^2} \tag{10}$$

where, r_1 is the inner radius, r_2 is the outer radius, and p is the boundary pressure. For this example, the inner radius of the cylinder is 0.5 m, the outer radius is 1 m and the boundary pressure is 10 kPa. The stress equations given in eq. 10 can be differentiated with respect to the inner radius to generate the sensitivities of the stresses. The sensitivities of the stresses with respect to the inner radius appear in eq. 11 and the second order sensitivities appear in eq. 12.

$$\frac{\partial \sigma_r}{\partial r_1} = 2\left[\frac{r_1 r_2^2 p}{(r_2^2 - r_1^2)^2}\left(\frac{r_2^2}{r^2} - 1\right)\right]$$

$$\frac{\partial \sigma_\theta}{\partial r_1} = -2\left[\frac{r_1 r_2^2 p}{(r_2^2 - r_1^2)^2}\left(\frac{r_2^2}{r^2} + 1\right)\right] \tag{11}$$

$$\frac{\partial^2 \sigma_r}{\partial r_1^2} = 2\left[\frac{(3r_1 + r_2^2) r_2^2 p}{(r_2^2 - r_1^2)^3}\left(\frac{r_2^2}{r^2} - 1\right)\right]$$

$$\frac{\partial^2 \sigma_\theta}{\partial r_1^2} = -2\left[\frac{(3r_1 + r_2^2) r_2^2 p}{(r_2^2 - r_1^2)^3}\left(\frac{r_2^2}{r^2} + 1\right)\right] \tag{12}$$

The problem was solved using four different meshes in order to examine the convergence of the error in the solution. The coarsest mesh contained 888 elements, and 1868 nodes, the next mesh contained 1792 elements and 3716 nodes, the third mesh contained 6184 elements and 12,608 nodes, and the finest mesh consisted of 25,124 elements and 50,724 nodes. The solutions for the radial and tangential stresses as calculated using the 6184 element mesh are shown in figure 1. The following norm was selected in order to compare the error in the four different mesh cases.

$$\|error\| = \frac{mean(|\sigma_{analytical} - \sigma_{numerical}|)}{mean(|\sigma_{analytical}|)} \tag{13}$$

The following formula was used to calculate the error plotted in the figures for this example.

$$error = \frac{\sigma_{analytical} - \sigma_{numerical}}{\max(|\sigma_{analytical}|)} \tag{14}$$

This formula was chosen so as to minimize the influence of large errors at locations where the solution is near zero, such as at the inner radius for the radial stress. Table 1 shows the norm of the error in both the radial and tangential stress solutions for each mesh case. This data is shown graphically in figure 2. It is seen that each successive mesh iteration reduces the error by approximately half an order of magnitude. The amount of computational time (wall time) needed to solve each mesh case is shown in table 2. Given the amount of computational time required for 25,124 element case and the fact that the complex sensitivity solutions will require three times more computation than the real valued case, the 6184 element case was used to generate the first, second and third order sensitivities. For each sensitivity method the step size or sampling radius was 0.001 which is approximately 1/30th of the average element edge length. CTSE and CD were both performed using as few sample points as possible, and FD was performed using 6 sample points. The norm of the error in the sensitivities appears in table 3. The error in the first order sensitivities of the radial stress over the entire domain appear in figure 3, and the error in the second order sensitivities of the radial stress appear in figure 4. These figures show only very slight differences between the three methods. It is also seen that along the inner circumference of the cylinder the error is very large. This is due to the fact that the sensitivity cannot be accurately calculated on the sampled surface itself, because the boundary conditions require the solution to be fixed at the inner surface.

The norm of the error in each case is mostly independent of the method selected. This is especially true for the first order sensitivities. The third order sensitivities show some small differences between the methods, with CTSE having the highest accuracy. The fact that the error is similar between all three methods points to the fact that the error in the solution is dominating any errors arising from the differentiation methods themselves. This is seen by looking at the first and second order sensitivities of the radial stress as a function of the number of elements (table 4). It is quickly seen that each additional mesh refinement increases the accuracy of the method. This reduction in the errors of the sensitivities is similar to the reduction in the error of the solution due to further mesh refinement seen in table 1.

The error in the first and second order sensitivities of the radial stress calculated using three different step sizes are shown in table 5. It is seen that changing the step size does not have much effect on the accuracy of the sensitivity. This is a further indicator that the accuracy of the solution is limiting the accuracy of the sensitivities, not the accuracy of the numerical differentiation methods. One exception is the second order sensitivity at the smallest step size, 0.0001 or 1/300th of the average element edge length. At this step size each method produces sensitivities that are less accurate than those calculated with a larger step size. This indicates that the machine round-off error associated with this step size may be similar in magnitude to the error due to the solution.

Numerical Example: Disc in Diametrical Compression

One of the classic tests in material analysis is the disc in diametrical compression [20]. In this test a circular disc is loaded in compression along its y-axis. The load is modeled as a point load. This loading and geometry generates a very nice uniform tensile stress along the xaxis of the specimen. It is thus useful in examining the tensile properties of a material without actually loading the specimen in tension. This test is also known as the indirect tensile test or the Brazil nut test.

The diametrical compression test has an analytical solution that can be derived through simple superposition. The solution of the stresses is seen in eq. 15 [28].

$$\sigma_x = \frac{-2P}{\pi}\left[\frac{(R-y)x^2}{\left(x^2+(R-y)^2\right)^2} + \frac{(R+y)x^2}{\left(x^2+(R+y)^2\right)^2} - \frac{1}{2R}\right]$$
$$\sigma_y = \frac{-2P}{\pi}\left[\frac{(R-y)^3}{\left(x^2+(R-y)^2\right)^2} + \frac{(R+y)^3}{\left(x^2+(R+y)^2\right)^2} - \frac{1}{2R}\right]$$
$$\tau_{xy} = \frac{2P}{\pi}\left[\frac{(R-y)^2 x}{\left(x^2+(R-y)^2\right)^2} + \frac{(R+y)^2 x}{\left(x^2+(R+y)^2\right)^2}\right] \quad (15)$$

In these equations, P is the magnitude of the point load, R is the radius of the disc, and x and y specify the location at which the stress is calculated, with the point (0,0) located at the center of the disc. The analytic solutions of eq. 15 can be differentiated to yield the sensitivities with respect to the radius of the disc. The equations for the first two sensitivities of the normal stress in the x-direction with respect to the radius are

$$\frac{d\sigma_x}{dR} = \frac{2P}{\pi}\left[\frac{(3R^2-6Ry-x^2+3y^2)x^2}{\left(x^2+(R-y)^2\right)^3} + \frac{(3R^2+6Ry-x^2+3y^2)x^2}{\left(x^2+(R+y)^2\right)^3} + \frac{1}{2R^2}\right]$$
$$\frac{d^2\sigma_x}{dR^2} = \frac{-2P}{\pi}\left[\frac{12(R-y)(R^2-2Ry-x^2+y^2)x^2}{(R^2-2Ry+x^2+y^2)^4} + \frac{12(R+y)(R^2+2Ry-x^2+y^2)x^2}{(R^2+2Ry+x^2+y^2)^4} - \frac{1}{R^3}\right] \quad (16)$$

The closed-form solutions for the sensitivities make this problem another excellent choice for exploring the use of the complex variable sensitivity methods.

The diametrical compression test model was solved using three different meshes, with a coarse mesh consisting of 1148 elements and 2357 nodes, a moderately refined mesh of 2502 elements and 5093 nodes and a fine mesh with 8,374 elements and 16,909 nodes. The solution for the stresses as calculated using the fine mesh appears in figure 5. It should be noted that for each figure in this example, no solution is plotted for the elements that share the node where the load is applied. This is due to the fact that the stress on this node would be infinite. The error norm used for this example is the same as for the thick walled cylinder example. The formula for the error plotted in the figures is.

$$error = \left| \frac{\sigma_{analytical} - \sigma_{numerical}}{\sigma_{analytical}} \right| \tag{17}$$

Table 6 shows the error for the three stresses for the three different mesh sizes. As before, it is seen that each successive mesh refinement results in a fairly large reduction in the error norm. This is seen visually in figure 6. The red areas are regions of high error. As the number of elements increases, it is seen that the total size of the red regions decreases significantly as the mesh is further refined. The computational time required to generate one solution appears in table 7. The errors in the sensitivities of the normal stress in the x-direction, calculated by each of the three methods are plotted in figure 7 (first order) and figure 8 (second order) for the fine mesh case. It is seen that there is again high error along the circumference of the disc due to boundary conditions. It is also noted that a few lines of high error form inside the discs. These lines represent regions where the analytical sensitivity is zero or near zero. Since the analytical sensitivity appears in the denominator of the error formula given in eq. 17, the error becomes very large when the analytical sensitivity tends towards zero. The norm of the error in the sensitivities are shown in table 8. It is seen that the norm of the error is much larger in this case. This is in keeping with the fact that the error norm of the solutions themselves are much higher than for the first example. This time it is seen that there is not a large dependence of the error norm on the number of elements. It is, however, seen that there is not much difference between the methods themselves. This points to the fact that the error in the derivatives is not due to the truncation error of the derivative methods, since the truncation error of CTSE and CD should be of the order $O(h^2)$ while the truncation error of FD should be $O(h^6)$.   The lack of dependence on the choice of method is further shown in table 9 where the norm of the errors is shown for the 2502 element mesh for three different step sizes, or sampling radii. Very little variation in the error is seen as a function of the step size, which is not the behavior of truncation error.

Conclusions

CTSE, FD, and CD can all be used to calculate shape sensitivities. This marks the first time that FD has been used for calculating finite element shape sensitivities. Unfortunately, For 2-D finite element problems using second order shape functions, the error in the model is greater than the error due to the truncation errors associated with the numerical differentiation methods. This means that the complex variable sensitivity methods do not offer extra accuracy compared to CD. If the model were made to be more accurate, such as with higher order basis functions or more elements, then FD and CTSE could offer improved accuracy. It has been shown that FD is capable of producing highly accurate sensitivities for functions with solutions that are accurate to machine precision [17]. The trade-off for the increased accuracy of FD is the requirement of several complex sample points. It takes three times more computational effort to generate a complex sample than a real valued sample. Thus for the problems described in this paper, FD is not a good choice for shape sensitivity calculations, due to the limited accuracy of the solution. CTSE requires only half of the number of sample points required by CD, thus CTSE only requires 1.5 times more computational effort than CD. This coupled with the fact that CTSE doesn't require changing the location of nodes in the complex plane means that it may still be a good choice for shape sensitivity problems.

One of the biggest problems with CD is that it requires the user to change the location of several nodes. Depending on the method chosen, this may lead to elements with poor aspect ratios, or complete remeshing of the domain, or both. This is especially true when the step size is rather large. Since CTSE does not require moving the nodes in the real plane, remeshing is not required, and there is less concern over the aspect ratio of the elements. Furthermore, if only the first order sensitivity is required, than the step size can be made very small without fear of increasing the round-off error. This may be very useful for problems in which external constraints may prohibit the domain boundary from moving.

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Tables

Table 1. The Norm of the Error in the Stress Solutions for Example 1

Number of Elements	Radial Stress	Tangential Stress
888	5.2510e-3	3.1852e-3
1792	2.6262e-3	1.7412e-3

6184	7.5747e-4	5.3097e-4
25124	1.8608e-4	1.3248e-4

Table 2. The Computational Time Required to Solve each Model for Example 1

Number of Elements	Time For Solution
888	8.49 s
1792	19.48 s
6184	130.76 s
25124	2799.77 s

Table 3. The Norm of the Error in the Sensitivity of the Stress to the Inner Radius for Example 1

Method	Radial Stress			Tangential Stress		
Order	1st	2nd	3rd	1st	2nd	3rd
CD	4.7265e-2	8.1689e-2	2.4292e-2	8.4286e-2	3.2213e-1	1.4765e-2
CTSE	4.7268e-2	8.1766e-2	2.4298e-2	8.2488e-2	3.1587e-1	1.4769e-2
FD	4.7268e-2	8.1671e-2	2.4295e-2	8.7353e-2	3.3258e-1	1.4769e-2

Table 4. The Norm of the Error in the First Order Sensitivity of the Radial Stress as a Function of the Number of Elements for Example 1

Number of Elements	CD		CTSE		FD	
Order	1st	2nd	1st	2nd	1st	2nd
888	1.2707e-1	2.0865e-1	1.2720e-1	2.0856e-1	1.2720e-1	2.0859e-1
1792	8.9792e-2	1.5315e-1	8.9924e-2	1.5214e-1	8.9934e-2	1.5248e-1
6184	4.7265e-2	8.1689e-2	4.7268e-2	8.1766e-2	4.7268e-2	8.1671e-2
25124	2.2981e-2	4.0049e-2	2.2983e-2	4.0088e-2	2.2983e-2	3.9894e-2

Table 5. The Norm of the Error in the First and Second Order Sensitivities of the Radial Stress as a Function of Step Size for Example 1

Step Size	CD		CTSE		FD	
Order	1st	2nd	1st	2nd	1st	2nd
.0001	4.7268e-2	8.2959e-2	4.7268e-2	1.0500e-1	4.7268e-2	9.2942e-2
.001	4.7265e-2	8.1689e-2	4.7268e-2	8.1766e-2	4.7268e-2	8.1671e-2
.01	4.7091e-2	8.2920e-2	4.7249e-2	8.1197e-2	4.7271e-2	8.1748e-2

Table 6. The Norm of the Error in the Stress Solutions for Example 2

Number of Elements	Norm of Error in Stress in X	Norm of Error in Stress in Y	Norm of Error in Shear Stress
1148	1.2615e-1	4.3792e-2	9.7171e-2

2502	8.6330e-2	3.2347e-2	6.1229e-2
8374	4.5780e-2	2.0857e-2	3.9533e-2

Table 7. The Computational Time Required to Solve each Model for Example 2

Number of Elements	Time For
1148	10.41 s
2502	27.82 s
8374	200.00 s

Table 8. The Norm of the Error in the First Order Sensitivity of the Normal Stress in X as a Function of the Number of Elements for Example 2

Number of Elements	CD		CTSE		FD	
Order	1st	2nd	1st	2nd	1st	2nd
1148	0.4681	0.6378	0.4681	0.6379	0.4681	0.6378
2502	0.4164	0.5673	0.4164	0.5675	0.4164	0.5674
8374	0.4267	0.6572	0.4270	0.6581	0.4270	0.6577

Table 9. The Norm of the Error in the First and Second Order Sensitivities of the Radial Stress as a Function of Step Size for Example 2

Step Size	CD		CTSE		FD	
Order	1st	2nd	1st	2nd	1st	2nd
.0001	0.4164	0.5679	0.4164	0.5726	0.4164	0.5688
.001	0.4164	0.5673	0.4164	0.5675	0.4164	0.5674
.01	0.4143	0.5609	0.4166	0.5707	0.4164	0.5674

Figures

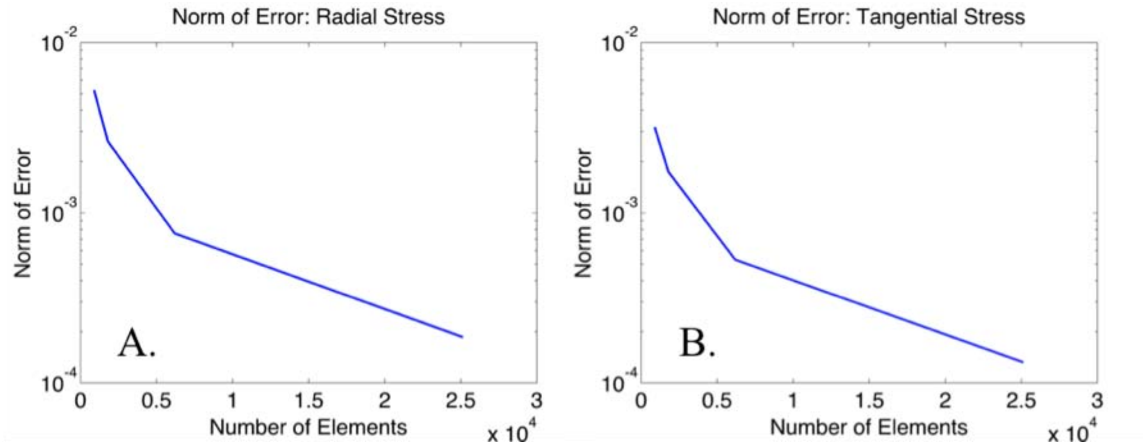

Figure 1. Convergence of the Error in the Radial and Tangential Stress Models for Example 1. *A. The norm of the error in the radial stress, B. The norm of the error in the Tangential Stress*

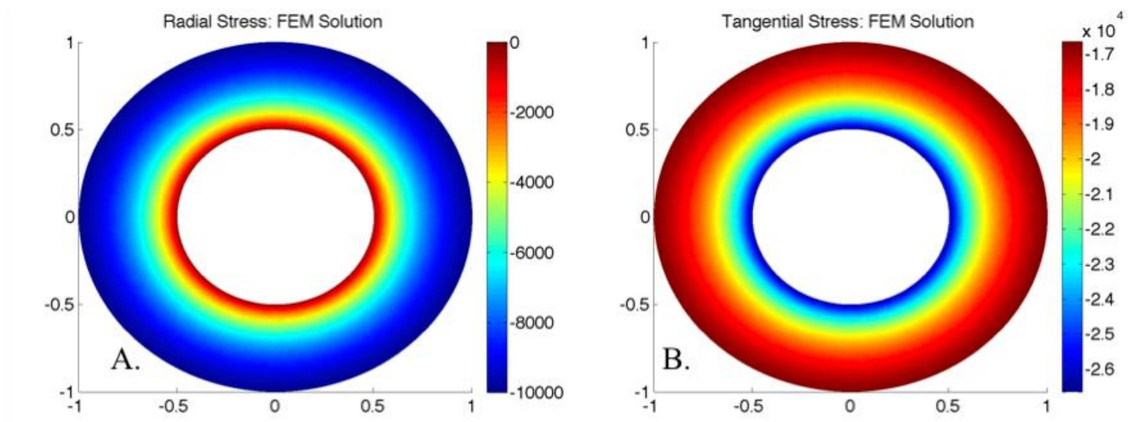

Figure 2. The Numerical Solution of the Radial and Tangential Stresses for Example 1. *A) The FEM solution for the radial stresses B) The FEM solution for the tangential stresses*

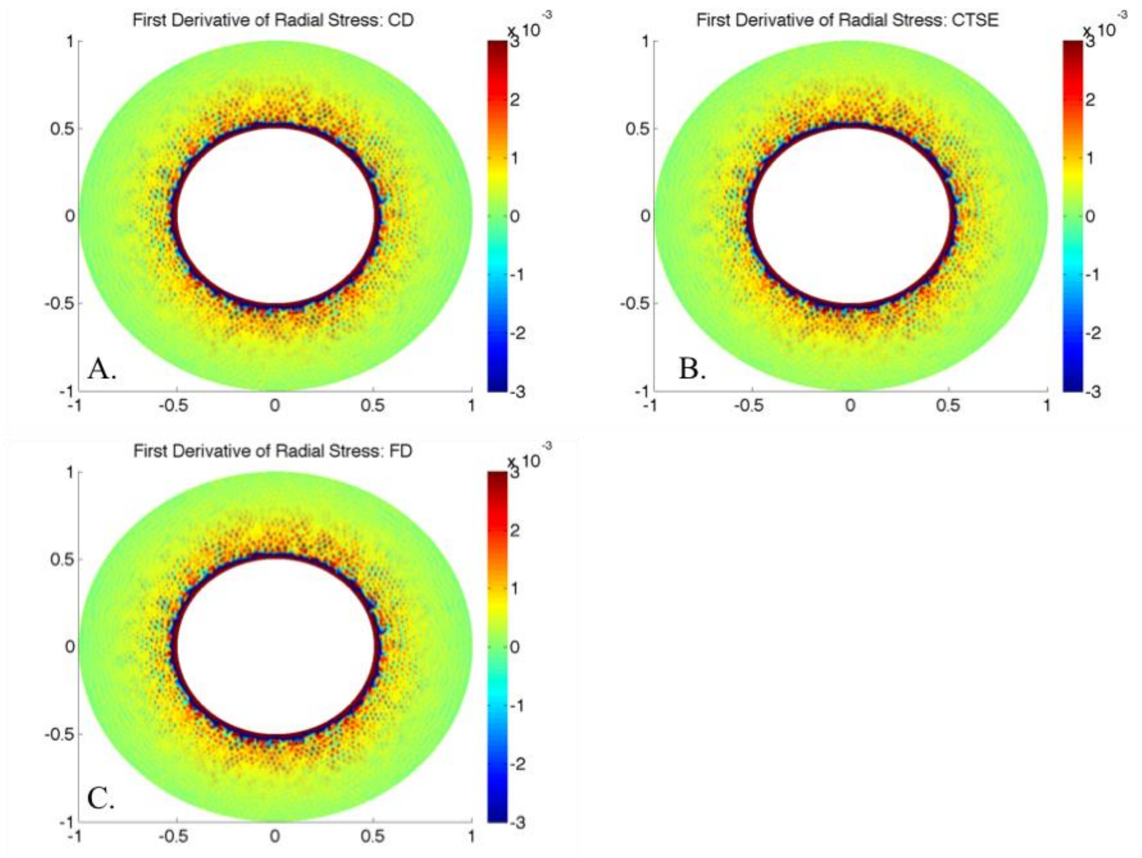

Figure 3. The Error in the First Order Sensitivity of the Radial Stress for Example 1. *A) Error in CD, B) error in CTSE, C) error in FD.*

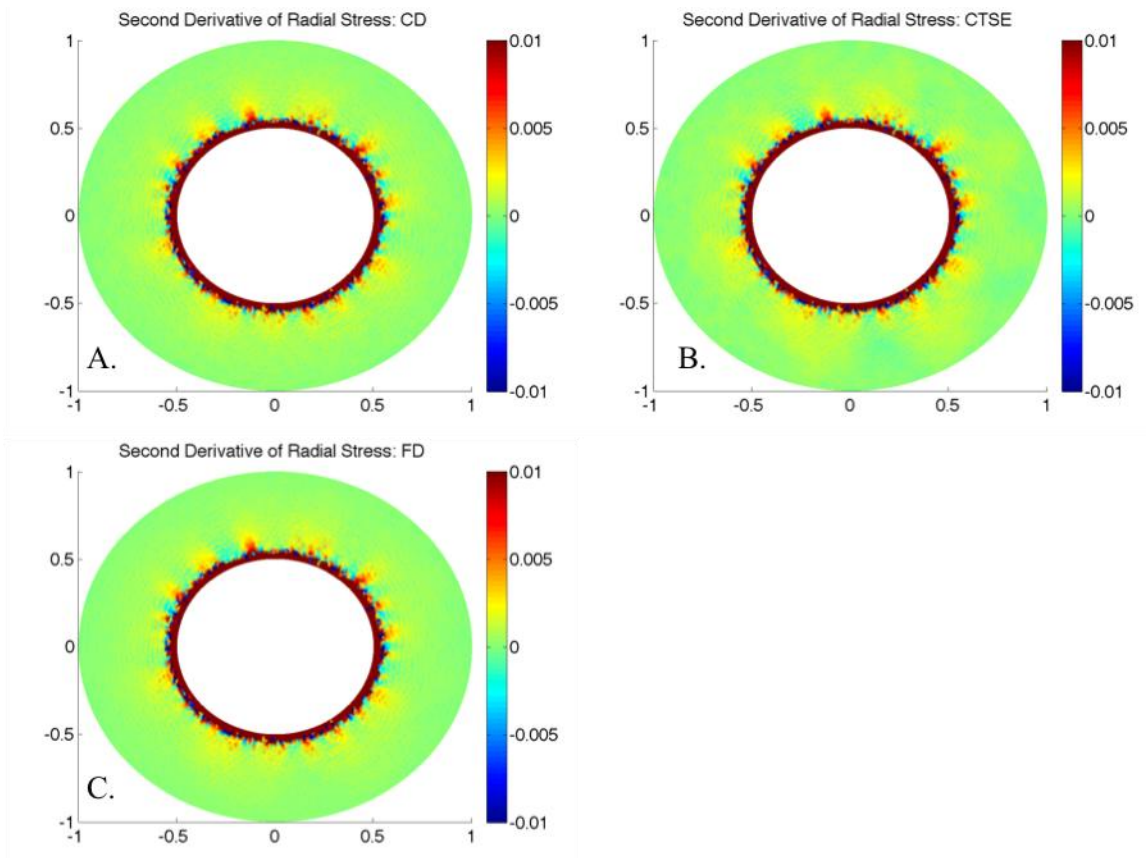

Figure 4. The Error in the Second Order Sensitivity of the Radial Stress to the Inner Radius for Example 1. *A) Error in CD, B) error in CTSE, C) error in FD.*

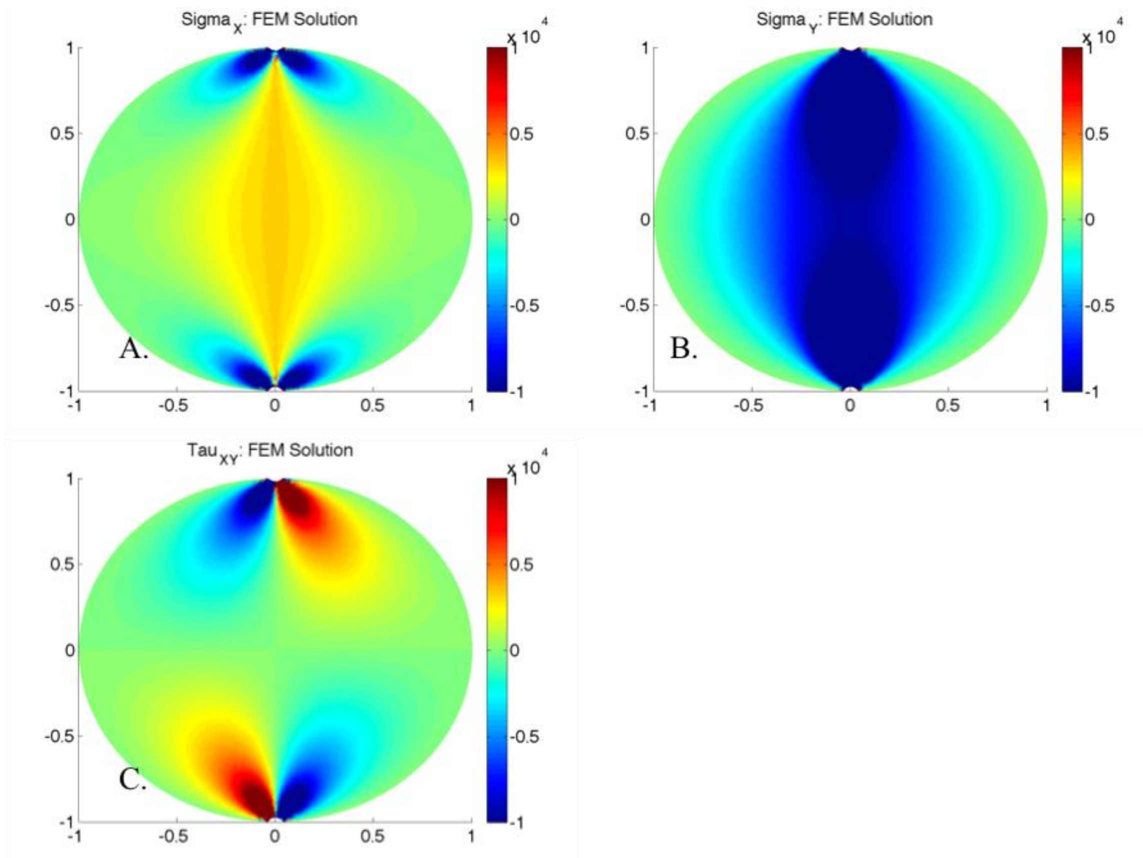

Figure 5. The Finite Element Solution for the Stresses in a Disc in Diametrical Compression. *A) The numerical solution for the stresses in the x-direction B) The numerical solution for the stresses in the y-direction C) The numerical solution for the shear stress*

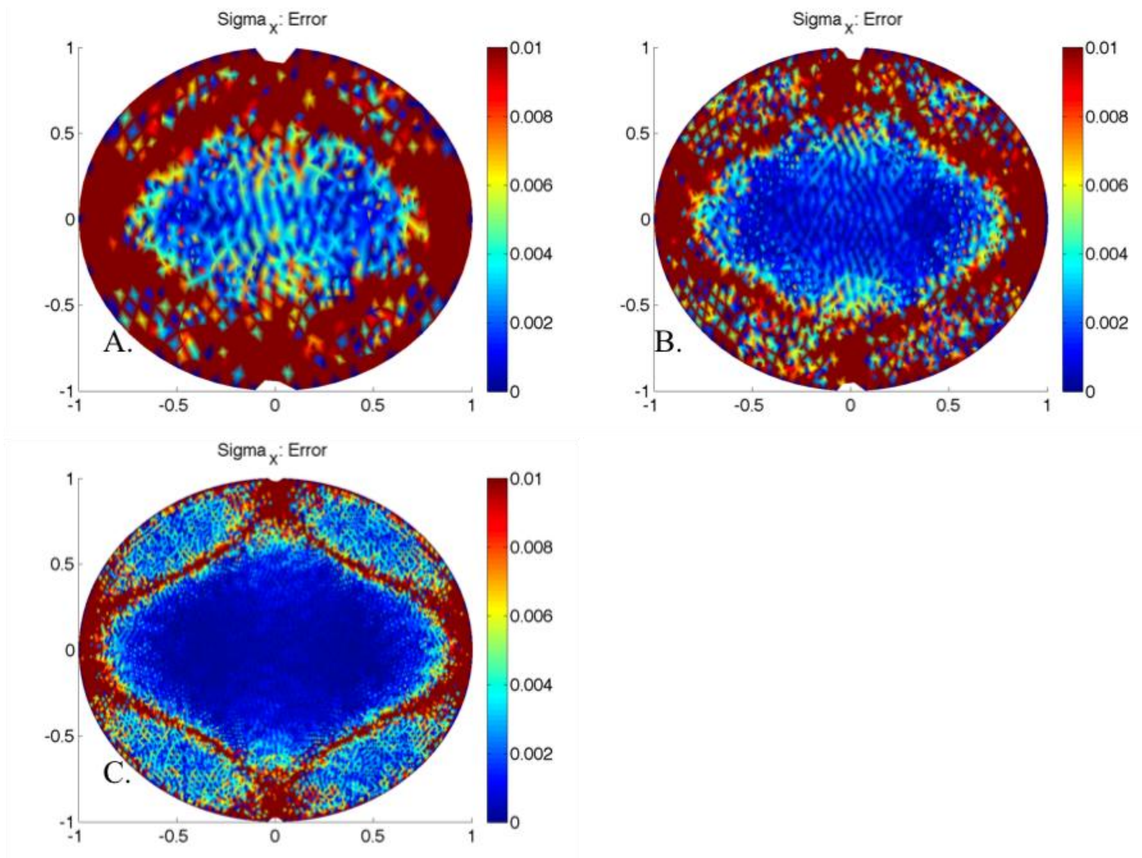

Figure 6. The Error in the Finite Element Method for Three Different Meshes for Example 2. *A) The error for the mesh with 1,148 elements and 2,357 nodes, B) The error for the mesh with 2,502 elements and 5,093 nodes, C) The error for the mesh with 8,374 elements and 16,909 nodes.*

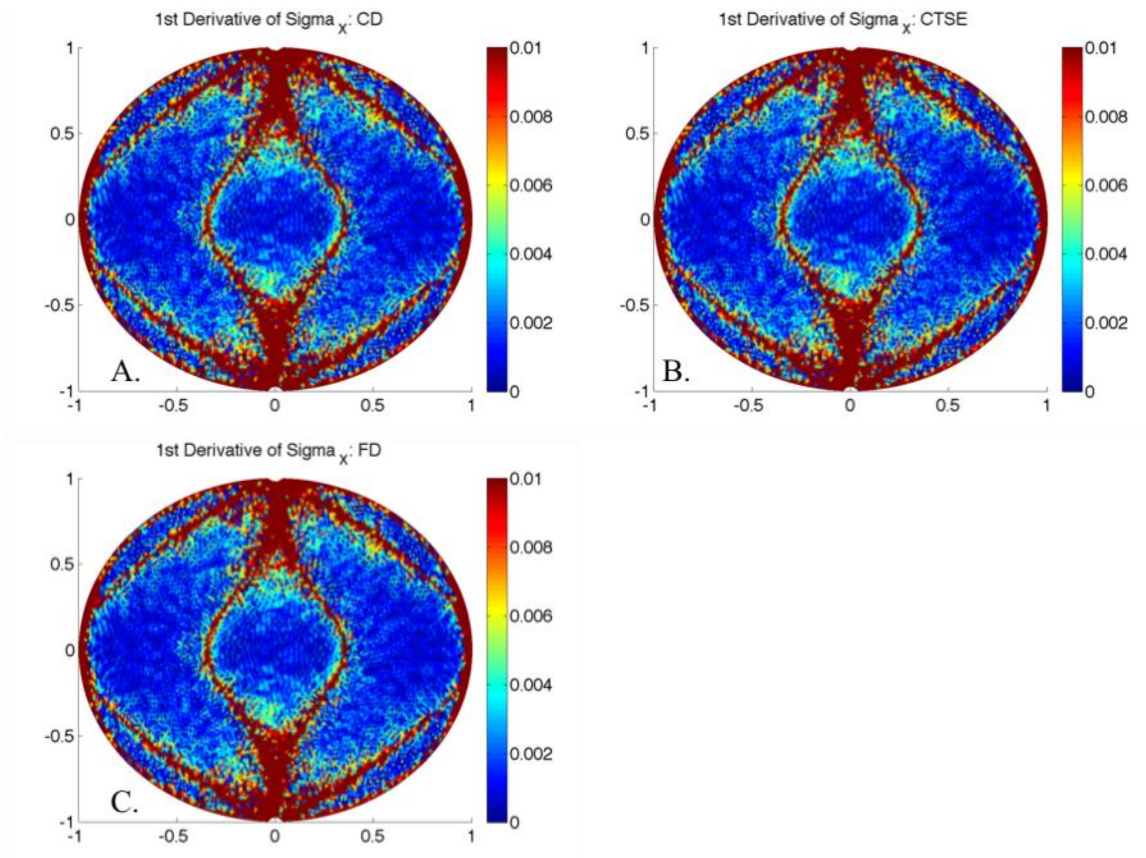

Figure 7. The Error in the First Order Sensitivity for Example 2. *A) Solution calculated by CD, B) Solution calculated by CTSE, C) Solution calculated by FD*

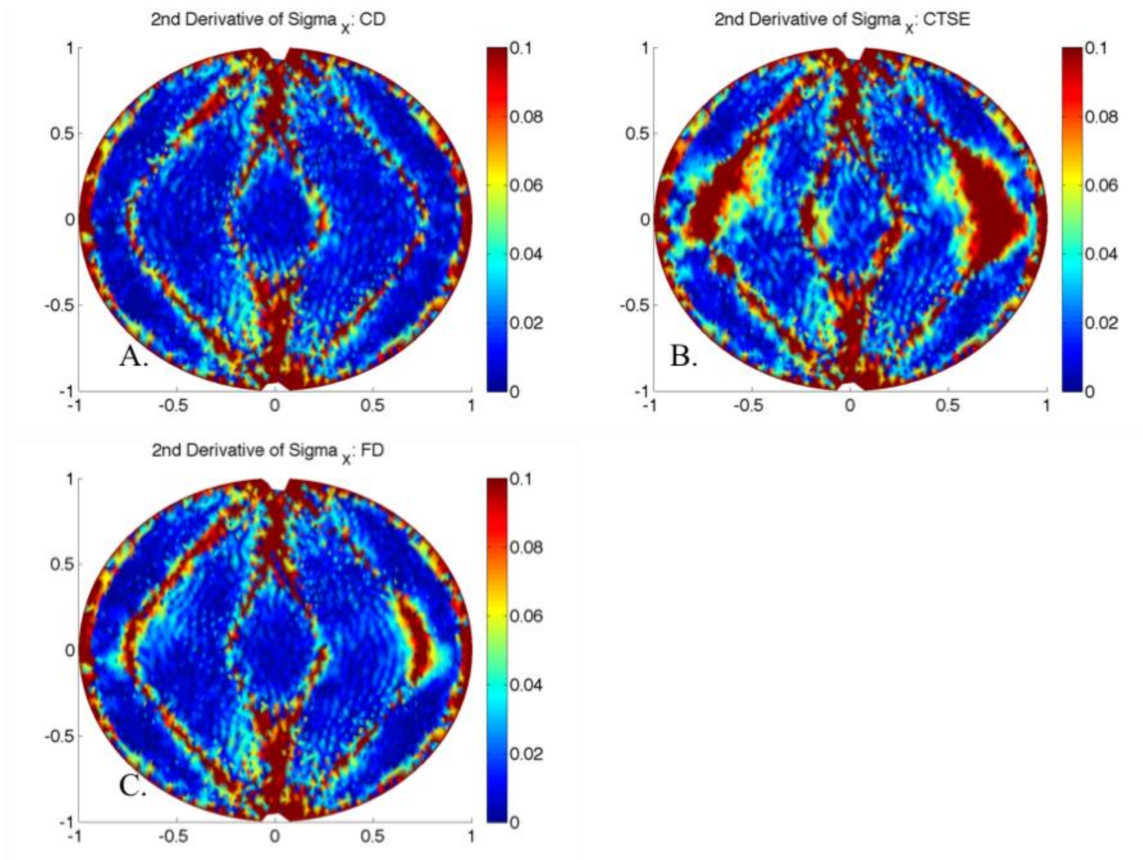

Figure 8. The Error in the Second Order Sensitivity for Example 2. *A) Solution calculated by CD, B) Solution calculated by CTSE, C) Solution calculated by FD*