The Statistics of GPS

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ABSTRACT

The Global Positioning System (GPS) is an extremely effective satellite-based system that broadcasts sufficient information for a user to determine time and position from any location on or near the Earth. The fundamental GPS measurement is the corrected time of the satellite clock relative to the receiver clock. This paper uses publicly available information to present a statistical analysis of the underlying timescale and clock performance, which can be largely presented without recourse to the many significant and interesting scientific corrections and parameterized models that could or must be applied to the data

Keywords: Allan, Clock, Frequency, GPS, Hadamard, Linear Quadratic Gaussian, Time

1. INTRODUCTION

The international standard for time is Coordinated Universal Time (UTC), which is published monthly in the Circular T by the International Bureau of Weights and Measures (BIPM)[1]. UTC is realized in real-time by the master clocks of each participating laboratory k, and designated UTC(k). The U.S. Naval Observatory's (USNO) realization of time is designated UTC(USNO). GPS broadcasts corrections so that all of its clocks can provide a predictive realization of UTC(USNO).

The GPS corrections are obtained with the use of data measured from what are now 11 monitor sites distributed fairly evenly in longitude, and located in the non-arctic latitudes. Each monitor site records the difference between the time broadcast on each satellite's signals with the time of its local clock. These differences are corrected for many effects beyond the scope of this paper (such as leap seconds which are intermittently inserted into UTC to account for the rotation of the Earth[2]), and used by the GPS Master Control Station as input to a Kalman filter[3] which models and predicts satellite clock performance in discrete 15-minute steps[4]. Measurements from the USNO are also applied to steer an implicit ensemble mean to approximate a prediction of UTC(USNO)[5].

The clock model applied by the GPS Master Control Station Kalman Filter (MCSKF) considers the phase (time can be considered to be the phase of a sinusoidal oscillation), frequency, and frequency derivative (frequency drift) to be parameters, and therefore this paper will first describe the Allan and Hadamard statistics[6], which are appropriate under the circumstances. The paper will then describe general considerations for a timescale, and provide details of how the GPS composite clock, whose realization is termed GPS Time, operates. There are many ways the GPS composite clock could be steered to UTC(USNO), and this paper will show one way to change the current algorithm.

2. THE ALLAN AND HADAMARD DEVIATIONS AND VARIANCES

Statistical measures used in precise timekeeping are nonstandard in other fields. Fourier techniques are rarely used because periodic terms, such as diurnal cycles, are frequently minimized through environmental isolation. Since no clock property is statistically stationary over long averaging or sampling times, statistical measures that depend upon differences over finite intervals of length τ are preferred. The Allan deviation is the most commonly used statistical measure, which we define in this section along with the Hadamard deviation.

To define these statistics, we denote the time reading of a clock by $x_k(t_i)$ where t_i is the time the clock is measured. The t_i are assumed to be measured at equal discrete intervals τ. The average frequency y_i of a clock between t_i and t_{i+1} is given by:

$$y_i = (x_{i+1}-x_i)/\tau \qquad (1)$$

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **2007**	2. REPORT TYPE	3. DATES COVERED **00-00-2007 to 00-00-2007**
4. TITLE AND SUBTITLE **The Statistics of GPS**		5a. CONTRACT NUMBER
		5b. GRANT NUMBER
		5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER
		5e. TASK NUMBER
		5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **U.S. Naval Observatory,3450 Massachusetts Ave, NW,Washington,DC,20392**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S)
		11. SPONSOR/MONITOR'S REPORT NUMBER(S)
12. DISTRIBUTION/AVAILABILITY STATEMENT **Approved for public release; distribution unlimited**		
13. SUPPLEMENTARY NOTES		
14. ABSTRACT		
15. SUBJECT TERMS		

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**8**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

The Allan deviation, denoted $\sigma_y(\tau)$, is the square root of the Allan variance, $\sigma^2_y(\tau)$, which is a scaled mean square of the second differences of the clock time readings, or the first difference of the frequencies:

$$\sigma^2_y(\tau) = \langle (x_{i+2} - 2\,x_{i+1} + x_i)^2 \rangle /(2\,\tau^2) \qquad (2)$$

Note that this statistic is independent of the clock's initial time and frequency. The division by two normalizes the statistic so the Allan deviation would equal the RMS of the clock that is characterized by white Gaussian noise.

The Hadamard deviation, denoted ${}_H\sigma_y(\tau)$, is the square root of the Hadamard variance, ${}_H\sigma^2_y(\tau)$, which is a scaled mean square of the third difference of the time, or the second difference of the frequencies:

$${}_H\sigma^2_y(\tau) = \langle (x_{i+3} - 3\,x_{i+2} + 3x_{i+1} - x_i)^2 \rangle /(6\,\tau^2) \qquad (3)$$

Note that this statistic is independent of the clock's average time, frequency, and frequency drift. It is sensitive to any change in the frequency drift, or to any change in the time or frequency which cannot be characterized through an unchanging frequency offset or frequency drift. The reader will have already surmised that the motivation for this statistic was to characterize clocks with unpredictable initial and average values of the time, frequency, or frequency drift. These are the rubidium-based atomic clocks which are currently used in the GPS satellites[7].

3. CONSIDERATIONS IN GPS TIMESCALE FORMULATION

A timescale can be described as a weighted average of corrected clocks. Timescales can be useful, if not required, because there is no way to measure absolute time[8], there is no perfect clock, and because only clock differences are measurable. For many applications, it is essential that the timescale be smoothly varying, and robust with respect to the addition, subtraction, or reweighing of individual clocks. In most cases, a real-time timescale is defined through extrapolation of a previously-determined timescale; the current value of the timescale is set equal the weighted average of the difference between the measured and predicted clock values.

Although the averaging process can easily create a timescale which at least equals the performance of an individual clock with regards to any statistic of interest, it is not possible to create a timescale that is simultaneously optimal with respect to every statistic. For example, the goal of the GPS system's composite clock is to create a timescale that is optimal for navigational solutions, but not necessarily to optimize the extraction of UTC(USNO). Therefore, a correction is broadcast with the message, so that UTC(USNO) can be extracted via a deterministic correction to GPS Time (Figure 1)[5]. The compromises involving in steering GPS time will be discussed in part V.

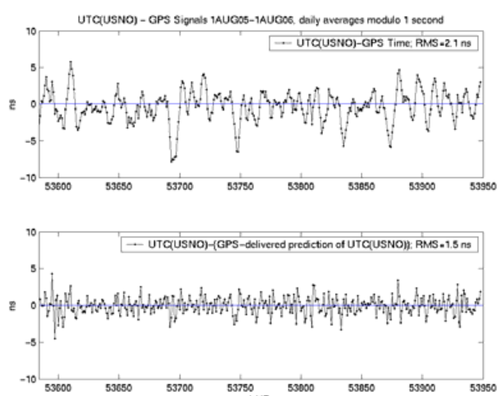

Figure 1. Upper plot is the daily average of UTC(USNO)-GPS Time, as measured at the USNO. Lower plot is the difference between UTC(USNO) and GPS's delivered prediction of UTC(USNO). Data are in nanoseconds, and MJD is the number of days since November 18, 1858.

The question of how to weight clocks in a timescale would be easily handled in the case of stationary Gaussian statistics, but in practice is complicated by the non-Gaussian, non-stationary, and not always measurable statistical nature of clocks[9]. Even a phenomenological approach that characterizes a clock's statistics against the timescale requires special care because a clock's contribution to the timescale has the effect of lowering the difference between the clock and the timescale. The stronger correlation would then increase that clock's weight in the timescale. Because every clock is 100% correlated with its contribution to the timescale, this would then lower the difference still more as the timescale is iterated or propagated. Ultimately, one clock would carry all the weight in the timescale. When one clock carries all the weight by design, the timescale is sometimes said to follow the "master clock approach". For a short time during the GPS development phase, the MCSKF selected a single ground station clock for this purpose. No parameters were used by the MCSKF to describe this one clock's state, and all other clock parameters effectively described the difference between the clock in question and the GPS master clock. The problems with that approach became very evident when that master clock failed, and subsequently the MCSKF adopted the composite clock formulation that is described next.

To describe each system clock, three parameters per clock are employed, which estimate its time, frequency, and frequency drift. The composite clock (cc) is an implicit weighted average of the corrected clocks (the average difference between the clock readings and their modeled readings). The cc is realized by the corrected value of every clock in the GPS system, but it is never actually computed as a numerical quantity because doing so is not needed to propagate the clock states in the Kalman formalism. The MCSKF propagates each clock state forward based only upon the measured differences of the satellite and monitor station clocks. As an additional step, all GPS satellite and ground clock frequency drift parameters are incremented equally on the basis of independent ground measurements of UTC(USNO)-GPS by the USNO. As shall be explained below, this steering keeps the cc stable and in line with UTC(USNO) on timescales of several days and more. Therefore estimates of the process noise that are based upon long-term averaging of observations are relative to UTC(USNO) and thus insensitive to the weights (inverse of the squared uncertainties) applied by the GPS MCSKF. New clocks are introduced to the constellation with zero weight, and their state parameters are set to correct them to the cc. Once weighted, the weight of any new clock would increase as more measurements are applied until it reaches the natural level set by the system process noises, which are not frequently changed. Individual observations (of clock differences) perturb the two relevant clocks in opposite directions and in accordance with the clock weights, therefore GPS Time varies with the weighted average of its clock readings and the MCSKF solutions are relevant only insofar as they affect the clock weights[4,10].

Since UTC is external to the GPS clocks, steering GPS Time to UTC(USNO) is essential. This is achieved through acceleration; the frequency drift parameter of every clock is adjusted by either 0 or $\pm 1.0 \cdot 10^{-19}$/s. A one-sentence summary of the algorithm is that the acceleration is set opposite to the slope in GPS-UTC(USNO), except that if doing so constantly thereafter would result in the extrapolation of the accelerated GPS-UTC(USNO) never becoming zero, in which case the acceleration has the same sign as the slope[11]. Figure 2 illustrates the process.

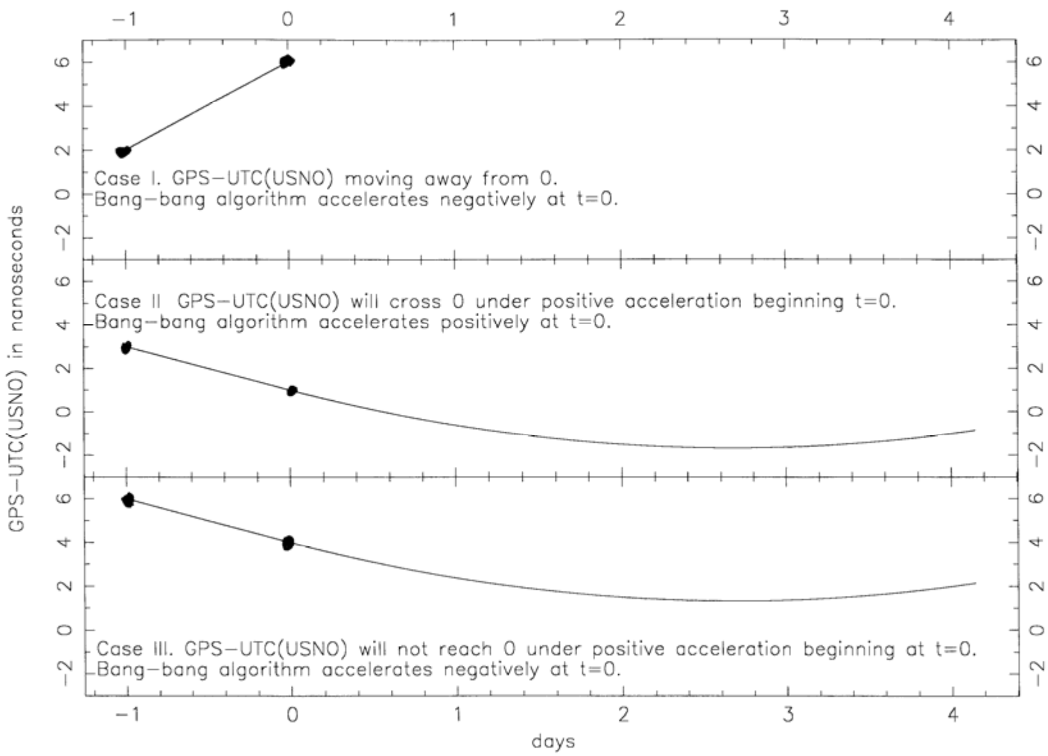

Figure 2. Pictorial display of GPS steering under three sample circumstances. Day 0 is the time of the steer computation, and day -1 is the previous day.

This steering in acceleration is often termed bang-bang, because it is quantized as on or off. The acceleration-based steering creates a tendency for GPS timescale to develop the oscillations evident in the upper half of Figure 1. Historically, the oscillations in GPS Time can be described as quasi-periodic with a typical duration of 20 days. One motivation for selecting this form of steering would be to attempt to minimize the subdaily variations in GPS Time, whose realization in the corrected satellite clocks is dependent upon uploads that are frequently delivered no more rapidly than once a day per satellite. A situation frequently exists wherein the receiver is simultaneously observing satellites whose broadcast time corrections, including the applied steering, are up to one day out-of-date compared to the current state values of the MCSKF. Since GPS Time is designed for position solutions that require all satellites to be on a common time, but not necessarily the correct time, the important thing is to optimize stability on the shortest averaging times. The accompanying long-term oscillations were deemed acceptable because corrections to recover UTC(USNO) are broadcast in the satellite navigation message so as to generate the data whose daily averages are shown in the lower plot of Figure 1.

IV. OBSERVED CLOCK AND GPS TIME STATISTICS

The International GNSS Service (IGS)[12] uses advanced carrier-phase techniques to compute the GPS satellite orbits, clocks, ionosphere corrections, and other parameters. As an associate analysis center of the IGS, the USNO provides clock solutions, which can be used to describe the difference between GPS satellite clocks and UTC(USNO). Figure 3 shows the observed Hadamard deviation of the GPS satellite clocks, using USNO-reduced data from a recent 20-day period, as reported to the IGS. The receiver was referenced to UTC(USNO), whose time reference is considerably more stable than any of the GPS satellite clocks.

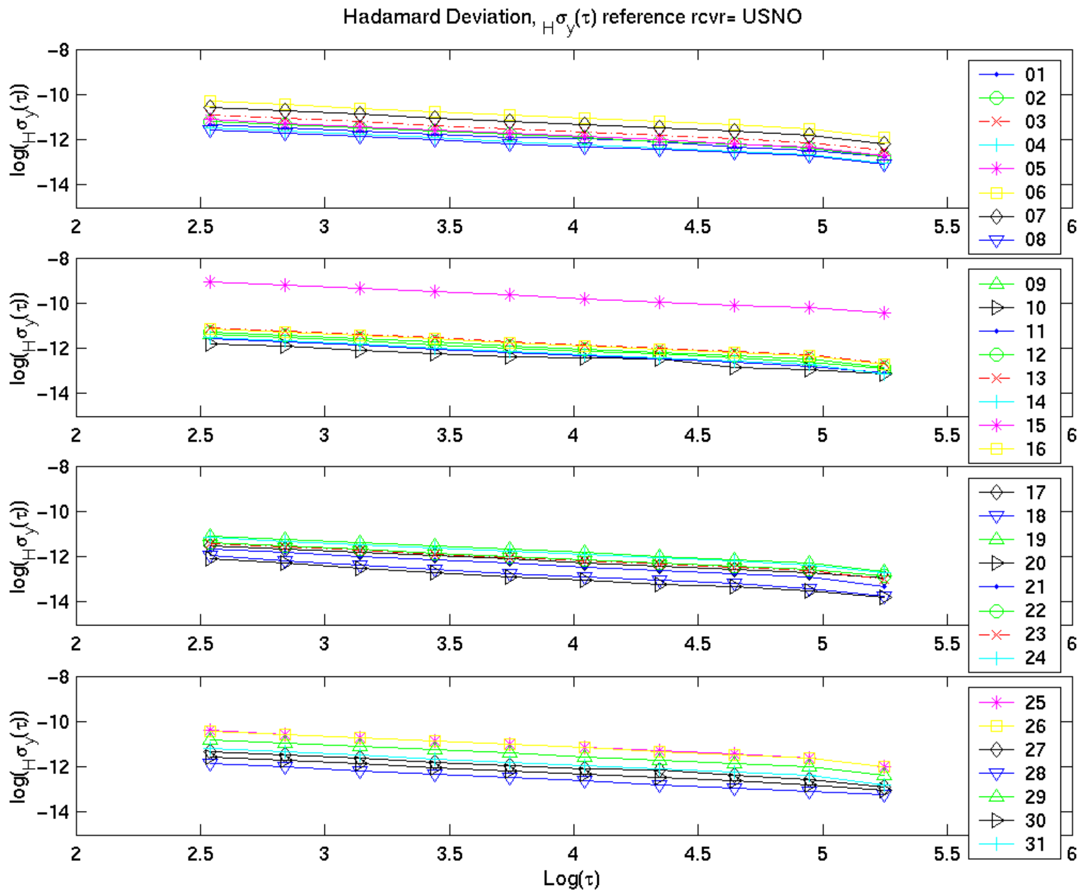

Figure 3. Statistical characterization of GPS satellite clocks as measured by the receiver USNO, which is referenced to UTC(USNO), MJD 54140-54160.

Most of the observed GPS satellite stabilities are identical to within a factor of 2, although a few satellites are more than an order of magnitude worse in stability. An algorithm which created GPS Time by averaging all satellites equally would generate results corrupted by the contribution of the few noisy satellites, such as SV15 in the figure. In the event of a satellite clock failure, the GPS timescale, as seen by the user, would show such effects before a bad-data message can be transmitted to the satellite. Here we discuss only timescales which apply weights according to some statistical measure of the clock stabilities, and ignore the data-integrity algorithms applied by the MCSKF. One could use the Hadamard deviations shown in Figure 3 to compute the expected Hadamard deviations of a timescale that weighs each clock by its inverse Hadamard variance. Such a timescale could base its weights on the statistics at any one of the intervals (τ), and therefore be optimal for that interval but not necessarily for other intervals, The computation of the expected timescale stability is a straightforward procedure analogous to more standard mathematics which would apply to a least squares evaluation involving the variances of a normal distribution. Figure 4 shows the expected timescale stabilities, and also shows the observed Hadamard deviation of GPS Time using the operational single-frequency SPS receiver at the USNO.

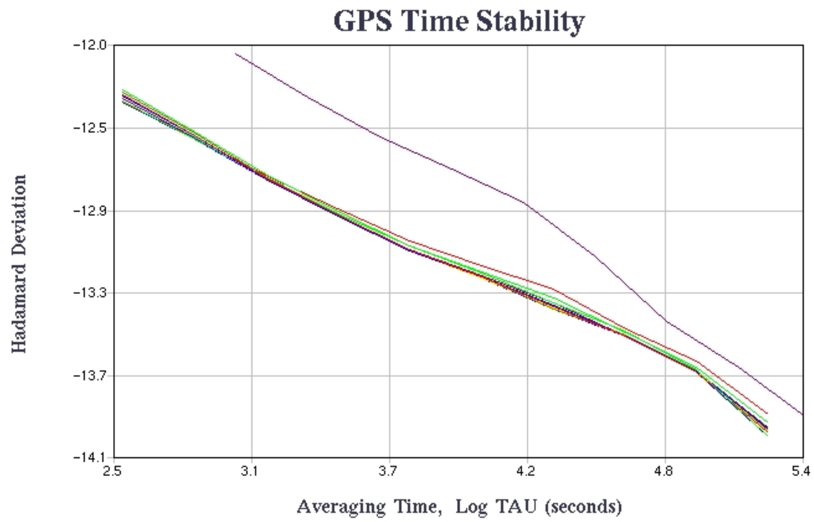

Figure 4. Stability of GPS Time (least negative curve) plotted against the theoretical limitations (lower plots). The discrepancies are explained in the text.

Because the individual curves of the clocks plotted in figure 3 rarely cross, the differences between the optimization schemes are barely noticeable. However, the measured value of GPS Time instability is considerably larger than would naively be expected by the clock stabilities, particularly over the smaller averaging times. This is due to the many neglected sources of measurement noise in the single-frequency GPS receiver, which does not use carrier-phase observations and relies upon ionosphere, orbit, and clock model parameters that are considerably less accurate when broadcast than the model parameters that are current inside the MCSKF. However, it is evident that the discrepancies become less as their averaging time approaches and exceeds a day, and this is strong evidence that the GPS composite clock's accuracy is not far from its theoretical limits at large averaging times.

V. STEERING GPS TIME

Most precise timing applications involve steering clocks in frequency, rather than the frequency drift (acceleration) steering used by GPS. In a typical situation, a frequency steer is applied that is described by the dot product of a 2 (or more) component gain function vector with a clock state offset vector that gives the time and the frequency offset (and perhaps higher order parameters) of the clock being steered. It is possible to compute an optimal gain function for any given "cost function", which assigns a relative importance of stability in time, stability in frequency, and the amount of control effort, by using a Linear Quadratic Gaussian (LQG) formalism[13-14]. It has been shown that LQG techniques can be used to generate a proportional steering method that would enable closer steering of GPS Time to UTC(USNO), for both the existing and expected future GPS constellation parameters[15]. Figure 5 and 6, taken from that work, show how the time determinations could be improved by proportional steering in a future GPS constellation.

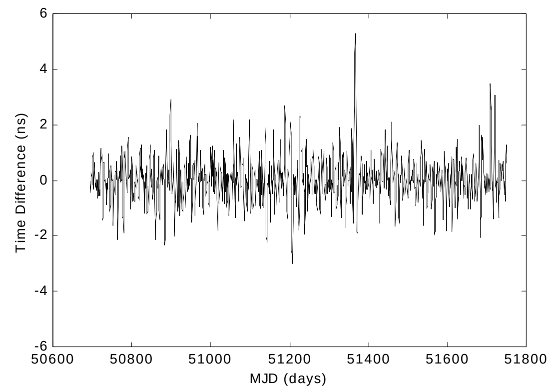

Figure 5. Simulated performance of a future GPS constellation using the current GPS acceleration-based steering algorithm

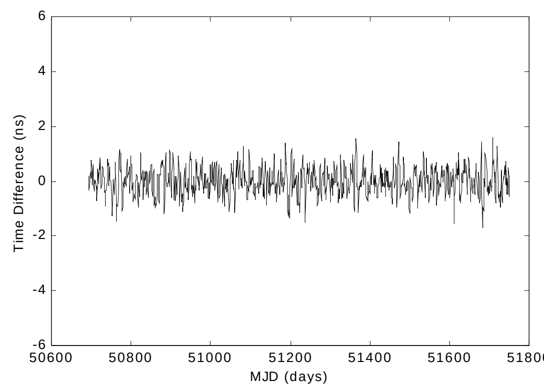

Figure 6. Simulated performance of cesium-based GPS constellation using a proportional steering algorithm selected to minimize frequency instabilities as opposed to time instabilities.

Figure 7 uses the Allan deviation to quantify the improvement[15]. This would be the appropriate statistic for a constellation using cesium atomic frequency standards, or any clocks that did not require estimation of the frequency drift.

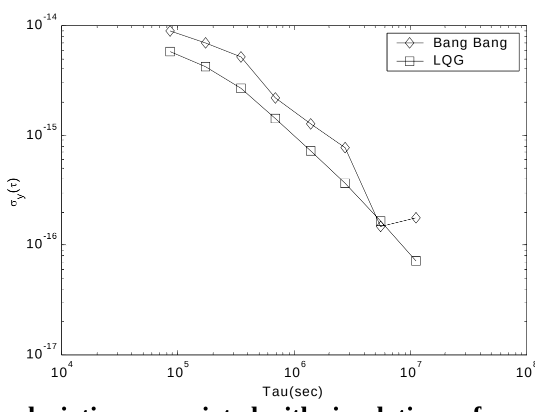

Figure 7. Allan deviations associated with simulations of previous two figures.

SUMMARY

The statistical properties of the GPS system can be characterized by standard precise timekeeping measures, such as the Allan and Hadamard deviations. Future work involving clock steering could prove fruitful, particularly as the GPS system and performance improve, and to improve interoperability with other GNSS systems.

ACKNOWLEGEMENTS

Thanks to the USNO Time Service Department for their dedicated efforts to ensure the USNO Master Clock provides the best possible time reference for GPS, and special thanks to Jim Skinner for a careful reading of the manuscript.

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