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**A SIMPLE NONPERIODIC RANDOM NUMBER GENERATOR - A
RECURSIVE MODEL FOR THE LOGISTIC MAP**

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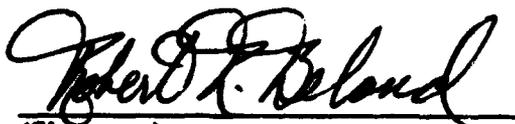


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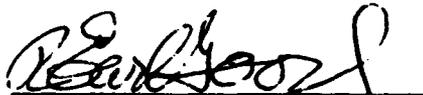


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13. ABSTRACT (Maximum 200 words) A nonperiodic random number generator which is based on the logistic equation is presented. A simple transformation which operates on the logistic variable and leads to a sequence of random numbers with a near-Gaussian distribution is described and discussed. The associated algorithm can be easily utilized in laboratory exercises, classroom demonstrations and software written for stochastic modelling purposes.				
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1 Introduction

In laboratory exercises for courses such as Statistical Mechanics and Thermodynamics, it is highly likely that undergraduate students will be required to write or use computer programs which utilize random number generators, ie to model the behavior of a particle undergoing Brownian motion. Although the idea that deterministic operations can produce sequences of numbers with random properties is paradoxical, it forms the theoretical foundation for all algorithmic random number generators. Most system-supplied random number generators are based on the linear congruential sequence (LCS) method[1][2]. This popular approach is limited, however, by the fact that the algorithm has a finite period, ie a cycle of numbers will eventually repeat. In this article, we present a simple, aperiodic, nonlinear random number generator which is based on the logistic equation. We describe a well-known transformation of the logistic variable for producing a sequence of random numbers with a uniform distribution, and propose a new method for generating a sequence with a near-Gaussian distribution.

2 The Logistic Equation

The logistic equation or map is given by the expression

$$x_{n+1} = 4\lambda x_n(1 - x_n), \quad (1)$$

where $0 < \lambda \leq 1$. This nonlinear difference equation, which maps the unit interval into itself, is the simplest example of a system capable of exhibiting chaotic behavior. It has been used to model such diverse phenomena as fluid turbulence, the evolution of biological populations, and the fluctuation of economic prices[4]. An excellent review on this subject was offered by May[5].

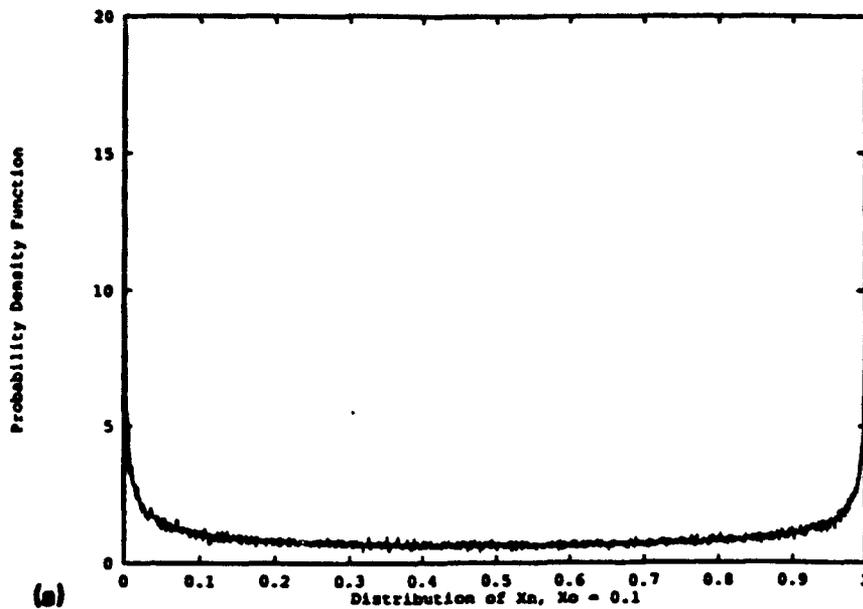
Ulam and von Neumann[6] studied the logistic equation with $\lambda = 1$. They demonstrated that iterates of the function generated a sequence of random numbers on the interval (0,1) with a continuous probability density P_x given by:

$$P_x = \frac{1}{\pi\sqrt{x(1-x)}} \quad (2)$$

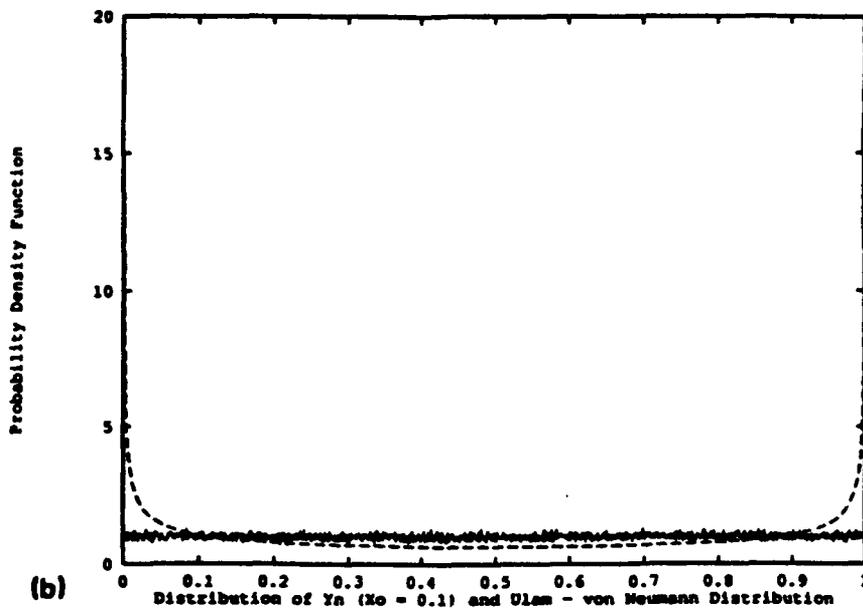
Theoretical and computer-generated results for the probability density of the logistic map are given in figure 1. Ulam and von Neumann also noted that by defining a new variable y_n as:

$$y_n = (2/\pi)\sin^{-1}(\sqrt{x_n}) \quad (3)$$

one could generate from the logistic variable a sequence of random numbers $\{y_n\}$ which is uniformly distributed on the interval (0,1)[7], Figure 1b.



(a)



(b)

FIG. 1. (a) Histogram with 1000 intervals of 10^5 iterates of the logistic equation with $x_0 = 0.1$. (b) Histogram with 1000 intervals of 10^5 iterates of y_n , the Ulam-von Neumann uniformly distributed random number generator given by Eq. (3), with $x_0 = 0.1$. The theoretical probability density for the logistic equation is superimposed as a dashed line.

3 The Logit Transformation

Given a sequence of uniformly distributed random numbers, it is possible to utilize a number of well-established transformations or techniques, i.e. the Box-Muller method, to produce a sequence of random numbers with a normal or Gaussian distribution. Detailed discussions on this topic can be found in Knuth[8] and Devroye[9]. In the present paper, we describe a transformation which operates directly on the logistic variable and generates a sequence of random numbers with near-gaussian distribution. The transformation, which is known in statistics as the *logit*, is given by the expression:

$$z = \ln \left(\frac{x}{1-x} \right). \quad (4)$$

For the sake of completeness, it is important to point out that this transformation leads to the standard logistic density, $Q_x = e^z / (1 + e^z)^2$, when it is applied to a uniformly distributed sequence of random numbers[9].

The probability density P_z for the logit transform of the logistic variable can be determined from the following relation:

$$P_x dx = P_z dz, \quad (5)$$

which can be rewritten as:

$$P_z = P_x \left| \frac{dx}{dz} \right|. \quad (6)$$

It can be shown from (4) that:

$$\frac{dx}{dz} = x(1-x). \quad (7)$$

Thus, combining (2), (6), and (7), one obtains:

$$P_z = \frac{x(1-x)}{\pi \sqrt{x(1-x)}} = \frac{1}{\pi} \sqrt{x(1-x)}. \quad (8)$$

Again, proceeding from (4), it can be shown that:

$$e^z = \frac{x}{(1-x)}, \quad (9)$$

and therefore:

$$x = \frac{e^z}{1 + e^z}. \quad (10)$$

Substitution of (10) into (8) thus leads to the desired probability density P_z :

$$P_z = \frac{1}{\pi} \left(\frac{e^{z/2}}{1 + e^z} \right) = \frac{1}{\pi(e^{z/2} + e^{-z/2})}. \quad (11)$$

The density P_z is remarkably similar to a Gaussian probability density (Figure 2).

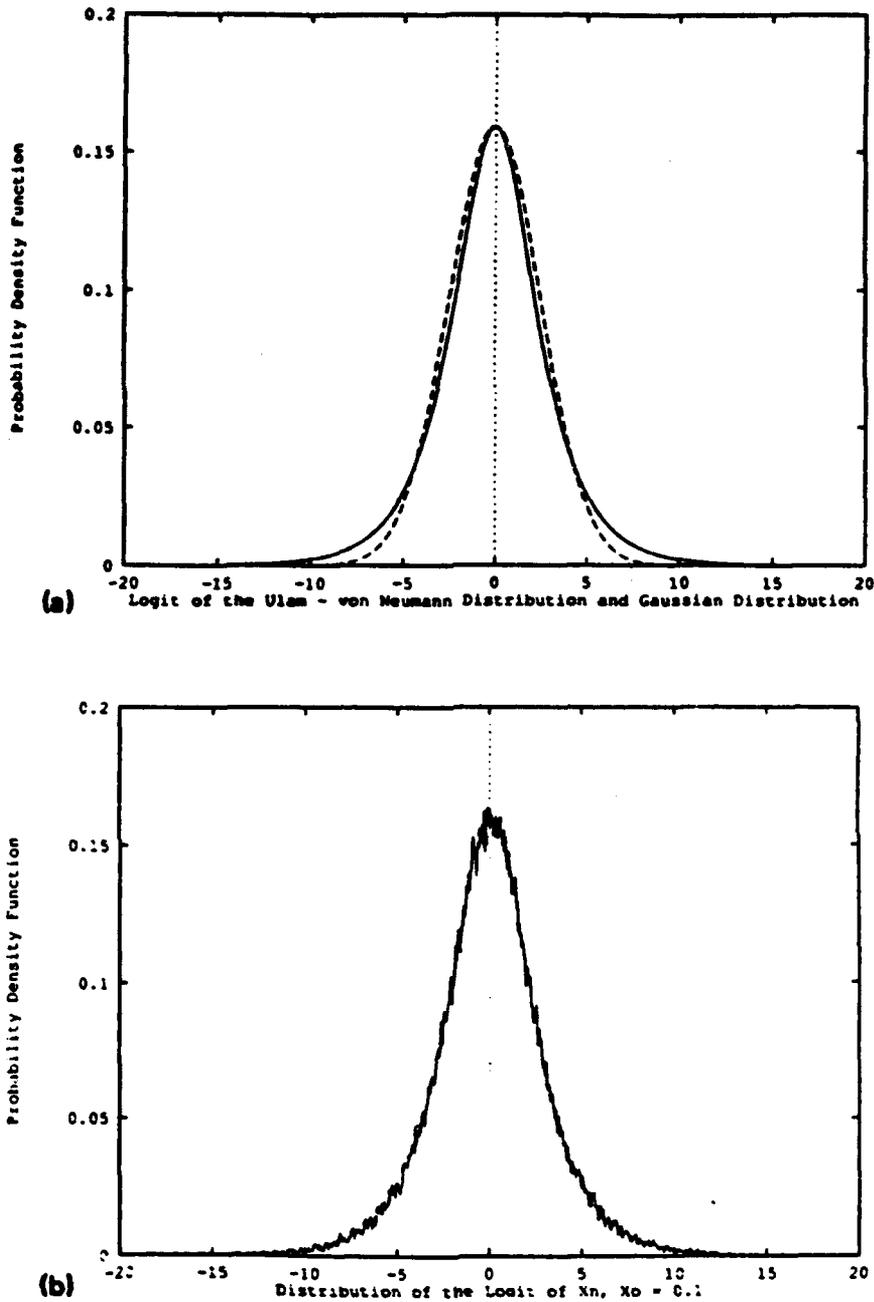


FIG. 2. (a) Plots of the theoretical probability density of the logit transform of the logistic variable (solid line) and a Gaussian probability density (dashed line). The comparison corresponds to the best least-squares fit of the parametric Gaussian, $\delta_\lambda(z) = e^{-z^2/\lambda^2}/\sqrt{\pi\lambda}$, to the logit transform of the logistic variable. (b) Histogram with 800 intervals of 10^5 iterates of the logit transform of the logistic variable with $x_0 = 0.1$.

4 Discussion

The present uniform and near-gaussian random number generators, which were developed from a priori theoretical consideration and based on simple non-linear deterministic equation, can be easily incorporated into software written for stochastic modelling purposes and Monte Carlo simulations. The associated algorithm has, in theory, an infinite period[10]-[12] and a correlation function which resembles a delta function[10][11]. It also has the advantage that it gives the n^{th} number directly without iteration. From a practical standpoint, it is important to point out, however, that the logistic equation has a number of unstable stationary points on the unit interval, i.e. $x_n = 0, 0.25, 0.5, 0.75, 1$. Thus, a small number of initial values (x_0) for the algorithm, i.e. those corresponding to or leading to the unstable stationary points, must be avoided when implementing the aforementioned random number generators. Nonetheless, from a pedagogical standpoint, these straight-forward exercises can be used in the classroom and laboratory to demonstrate the intimate relationships between many of the fundamental concepts underlying random-number generators, probability theory, and chaotic dynamics. We observe that, since z_n is a random number with a quasi-gaussian distribution, it can be represented by a very simple recursive model i.e.: $z_n = \epsilon_n$ where ϵ_n is quasi-gaussian subject to the distribution given by equation 11.

References

- [1] W.H Press et al., *Numerical Recipes* (Cambridge University Press, Cambridge, 1986)
- [2] P. Mennen, "DSP chips can produce random numbers using proven algorithm", *Engineering design News* **36**, 141-146, January 21, 1991.
- [3] M.J. Feigenbaum, "Universal behavior in nonlinear systems", *Physica* **7D**, 16-39, 1973.
- [4] R.V. Jensen, "Classical Chaos", *American Scientist* **75**, 168-181, 1987.
- [5] R.M. May, "Simple mathematical models with very complicated dynamics", *Nature* **261**, 459-467, 1976.
- [6] S.M. Ulam and J. von Neumann, "On combination of stochastic and deterministic processes", *Bull. Amer. Math. Soc.* **53**, 1120, 1947.
- [7] P.R. Stein and S.M. Ulam, "Non-linear transformation studies on electronic computers", *Rozprawy Matematyczne* **39**, 1-66, 1964.
- [8] D.E. Knuth, *Seminumerical algorithms*, 2nd ed. vol. 2 of *The Art of Computer Programming* (Addison-Wesley, Reading, MA, 1981).
- [9] L. Devroye, *Non-Uniform Random Variate Generation* (Springer-Verlag, New York, 1986).
- [10] S. Grossman and S. Thomae, "Invariant distribution and stationary correlation functions of one-dimensional discrete processes", *Z. Naturforsch.* **32a**, 1353-1363, 1977.
- [11] S. Thomae and S. Grossman, "Correlations and spectra of periodic chaos generated by the logistic parabola", *J. Stat. Phys.* **26**, 485-504, 1981.
- [12] J.J. Kozak, M.K. Musho, and M.D. Hatlee, "Chaos, periodic chaos, and the random-walk problem", *Phys. Rev. Lett.* **49**, 1801-1804, 1982

Appendix

100 Years of Turbulence, an Overview, updated edition of invited paper, presented by Dr. G. v.H. Sandri at the MIT symposium on Computational Fluid Mechanics, 1984 (M. Murman and L. Morino, Ed.).

1 Introduction

Atmospheric flows have played a fundamental role in furthering our understanding of turbulence from the very beginning of the subject. Boussinesq introduced the famous concept of effective viscosity at the end of last century in analysing transport of mass, momentum and energy in turbulent atmospheres. In 1914 G.I. Taylor, widely recognised as the founder of modern turbulence science, built and used extensively a two-jointed vane to measure turbulent velocity fluctuations in the atmosphere. This occurred long before hot-wire and laser anemometry became known. Fifteen years later, he published his famous series of papers in the Proceedings of the Royal Society on the statistical approach to turbulence.

The next major advance in our understanding of turbulence again has come from meteorology with the discovery of chaotic behavior in the Benard problem (motivated by the atmospheric surface layer) by Ed Lorenz at MIT in 1960. This work opened the field now known as "chaos dynamics" which has found fruitful applications in innumerable areas of research. In a nutshell, chaos dynamics starts with rigorously deterministic governing equations and yields as output stochastic solutions. However, in spite of the very extensive literature in the late seventies and eighties, chaos dynamics has not been applied to understanding the free atmosphere.

Our understanding of turbulence has been based on extensive research carried out over the last 100 years along two presumed distinct lines of thinking : the deterministic approach geared to the solution of the Navier-Stokes equations of fluid mechanics and the statistical approach geared to the analysis of averaged fluid equations supplemented by plausible statistical hypotheses. The deterministic approach and the statistical approach are now finding a unification through chaos dynamics in a spectacular development that in our opinion is a veritable scientific revolution. In chaos dynamics, purely deterministic equations yield stochastic solutions. In the overview of turbulence research we retrace briefly the two classical lines of thinking as well as outline the contemporary chaos dynamics approach.

2 Deterministic Approach

The classical deterministic theory was initiated by the analysis of linear stability of the fluid equations. The early work of Rayleigh was brought to full mathematical form in the Orr-Sommerfeld (Ref. 5) theory which has been perfected by Tollmein and Schlichting, and later by C.C. Lin and Chandrasekhar (Ref. 5) among other. Linear stability theory is by no means a closed chapter of fluid mechanics largely because the linear equations are of high order (e.g. third) and have non-constant coefficients (Ref. 63). Considerable efforts have been made in developing post-linear analysis and nonlinear stability. The results of this extremely difficult theory are very far from complete. In brief outline the stability theory

consists in separating the fluid variables into two parts

$$\vec{u} = \vec{u}_b + \vec{u}_p \quad (12)$$

the base flow variables describe a given flow, usually a steady flow (Ref. 3). The perturbation variables describe the departure from the base flow which are taken to be small compared to the base values in the linear stability theory. Substitution of the decomposition given in (1) into the Navier-Stokes equations, and, neglect of the second order (quadratic) terms in the perturbation variables, leads to the linear form of stability theory. Fourier model analysis in the unbounded direction is the standard procedure for solving the resulting linear equations. This analysis leads to a dispersion relation linking the wave vector of the disturbance added to the base flow and its frequency which is taken to be complex:

$$\omega = \omega(\vec{k}) = Re(\omega) + i Im(\omega) \quad (13)$$

Highly dispersive waves arise in this way. The unstable waves are identical to those with positive $\Im(\omega)$ (in the standard convention for the imaginary part of the frequency, $e^{-i\omega t}$). In boundary layer flow these waves are called Tollmien-Schlichting waves after the two theoreticians that calculated their properties (Ref. 63).

A major gap remains in understanding the actual onset of turbulence even when the stability boundary is known. In particular, the Reynolds numbers at which instabilities appear is much smaller than the Reynolds number at which turbulence (as measured for example through the skin friction coefficient) appears. The fact that the linear stability theory does not give a correct prediction of the critical Reynolds number for the onset of turbulence is responsible for the disrepute in which linear stability theory has been in the last half century. This explains, in part, why two world wars separate the theoretical prediction of incompressible waves in boundary layers from the experimental verification of the properties of these waves. The experiments were performed with great care and ingenuity by Schubauer and Skramstadt at a NACA facility that had a sufficiently clean wind tunnel not to mask the results with background turbulence. In order to bridge the gap mentioned in the previous paragraph, the theoretician Landau conjectured that new instabilities occur with increasing Reynolds number and that the accumulation of these instabilities would eventually lead to the onset of turbulence (Ref. 26). The conjecture led to many fruitful researches both theoretical and experimental. On the theoretical side, postlinear theory strove to determine the stability of the first perturbed flows. Such secondary flows are well known experimentally since the discovery by G. I. Taylor of the "rolls" in cylindrical Couette flow and they have been the subject of important and beautiful investigations by Gollub and Swiney, among others (Ref. 5). At present the state of the fluid past the second steady state (the wavy Taylor rolls) in cylindrical Couette flow is still unknown in spite of much effort. The theory is very challenging but very difficult also. The stability of the wavy Taylor rolls seems to be sensitively dependent on the aspect ratio of the Couette annulus. Chaos dynamics has however thrown a clear light on Landau's conjective by giving a precise mathematical model of the cascade of instabilities. The logistic map, in particular, provides such a rigorous

model. In the choice of coordinates that gives the logistic equation in the form

$$x_{n+1} = \lambda x_n(1 - x_n) \quad (14)$$

the onset of instability occurs at $\lambda = 1$ while the onset of turbulence, which is defined as the accumulation point of the bifurcations occurs at about 3.5. The logistic map has been shown by Ed Lorenz to arise, at least approximately, in the Benard problem and therefore we now possess both a sound mathematical model for the Landau conjecture and a physical basis for it (Ref. 33). It should be emphasized that there is no reason at present to believe that infinite cascading of instabilities (bifurcations in particular) is the only mechanism for the onset of turbulence. For example Ruelle and Takens proposed that only two successive instabilities occur in nature prior to the onset of turbulence. The theoretical status of their work is not very clear at present (Ref. 10). Also, explosive instabilities have been found in complex valued maps (Ref. 8).

3 Statistical Approach

The statistical approach to turbulence is now believed very widely to be valid, largely because measurements of fluid variables in a recognizably turbulent flow (e.g. a well developed one) show definite stochastic behavior (Refs. 2, 3, 38, 65, 68, 69, 72). The idea of a statistical approach started at the end of the last century with fundamental investigations of Reynolds, Poiseuille and Hagen on the flow of water in pipes. Reynolds recognized that when the ratio

$$Re = \frac{UL}{\nu} \quad (15)$$

exceeds a critical value, the fluid flow is turbulent. Reynolds called such flows "sinuous" and the statement $Re > Re_*$ he called "the criterion". G. I. Taylor formalized this approach very successfully (Ref. 66). The statistical theory of turbulence received great impulse during World War II when simultaneously and independently Kolmogoroff in Russia, Onsager in the USA and Heisenberg (Ref. 19) and von Weizaker in Germany reached the theoretical conclusion that a universal inertial range occurs in all turbulent flows with a power spectrum of velocity fluctuations satisfying a power law with exponent $-5/3$. Such a spectrum has been found in an important oceanographic experiment by Stewart and coworkers in the Discovery Channel in Canada. Atmospheric spectra have also been found to satisfy the inertial range and many laboratory experiments have exhibited inertial ranges.

3.1 Inertial Range Analysis

The tool behind Kolmogoroff's derivation of the 5/3 law (as it is now known) is dimensional analysis. The physical intuition is that in the inertial range of the turbulent spectrum (of

isotropic turbulence) only two parameters are relevant: the viscosity and the dissipation. The viscosity is a property of the material while the dissipation is a property of a specific flow. The dissipation is defined as the non-diffusive part of the viscous rate of change of the kinetic energy in the flow. It is, as a consequence of its definition, a non-negative quantity and measures precisely the rate at which fluid kinetic energy is transformed into heat energy by viscosity. The dimensional analysis of the energy spectrum, which is given in full detail in Batchelor's (Ref. 2) or Hinze's (Ref. 20) excellent books on turbulence, leads to the conclusion that, under the assumptions stated 1) the power spectrum has a $5/3$ decay law and 2) that the autocorrelation (of the longitudinal) velocity has a spatial decay which is a power law with index $2/3$. (This second result is discussed at some length in C.C. Lin's book (Ref. 31., also Ref. 68).) Landau pointed out, after the second world war, that the dissipation being a flow property rather than a material property, is subject to statistical fluctuations. The objection of Landau was followed up by simultaneous and independent work by Obukhov and Kolmogoroff with two papers in the Journal of Fluid Mechanics (Ref. 22, also Ref. 68). In these two papers the assumption is introduced that the turbulent fluctuations are log normal. A consequence of this and similar assumptions are still being investigated. Several (gentle) modifications of the $5/3$ law have been studied in the intervening years. Recent work at Cornell in this direction is very promising (Ref. 40). Experimental verifications of Kolmogoroff's law are numerous (Refs. 15-18, 47, 72).

3.2 Heisenberg and Related Energy Spectrum Models

A very important model of the evolution of the power spectrum was proposed by Heisenberg and von Weizsacker (Refs. 2, 19, 31). In this model a plausible mechanism is proposed to build the triple velocity correlation T (in wave number space) in terms of the energy spectrum alone. Since the energy spectrum is the Fourier transform of the two point velocity correlation, the Heisenberg model reduces the triple to the double velocity correlation by a (strong) physical assumption. The model was given an exact analytic solution by Bass and others (the solution of the Obukhov model was found in a paper by Milsaps). The results of major interest are two: 1) the Heisenberg model has an inertial range with the $5/3$ power law 2) the very tiny eddies decay with a precise stronger law. Experimental support for the Heisenberg tail has been found, although there is no substantial evidence for universality of the tail. The model of Heisenberg has influenced in a deep way the further evolution of analysis of turbulent flows since not only it generated many interesting alternative closures of the spectral equation (like that due to Obukhov mentioned above and a very general one due to von Karman (Refs. 2, 34) but also and more importantly it is the germ of the second order closure techniques of the sixties which are briefly summarized below. It is worth mentioning that important models that followed after Heisenberg's, were spearheaded by the "quasinormal" approximation suggested by Millionshchikov who reduced the fourth order correlation to the second, assuming that the relation be the same as if the underlying statics were gaussian (Refs. 42, 43). Substantial differences were found

between the quasinormal two point and the quasinormal two point and two time assumptions by both Heisenberg and Chandrasekhar. The direct interaction approximation (in several successive improvements) developed by Kraichnan and others can be considered a cousin to the Heisenberg and quasinormal models (Refs. 24, 25). While very interesting results have been obtained along the lines described above, it has been very difficult to extend the energy spectral models to non-homogeneous turbulence.

3.3 Second Order Closure

A different line for the analysis of turbulent transport was initiated by work carried out first at Cal Tech in the thirties by Chou (who is now the culture minister of the People's Republic of China). Chou's main idea was to obtain turbulent transport laws, analogous to the Fick, Navier-Stokes and Fourier laws for molecular transport. The idea was improved by a student of Heisenberg's, Rotta, in the fifties. This program finally received full attention when large scale computational solutions of many coupled non-linear partial differential equations became possible in the sixties as a consequence of the advent of fast computers. The two major variants of turbulent models (with space and time variables contrasted to wave number variables) that are in substantial use today are the so called $k - \epsilon$ model and several variations of the Reynold's stress closure (Refs. 6, 7, 15, 28, 30, 36, 39, 51-61). These models are of interest because on the basis of their governing equations, with appropriate boundary and initial conditions, it has been possible to obtain quantitative prediction on important examples of real flows such as boundary layer flows, jets and wakes where isotropy and homogeneity do not apply even approximately (Refs. 9, 12, 21, 23, 25, 46, 49, 50, 64, 70, 71, 73).

In the $k - \epsilon$ models, fluid equations are assumed simultaneously for the turbulent kinetic energy and for the dissipation, triple and higher correlations being modeled along the Chou-Rotta line of reasoning. The physical intuition underlying the modeling is made clear when one thinks of kinetic energy and scale (typical eddy size) rather than in terms of kinetic energy and dissipation. The relations among quantities is given by the equations

$$k = q^2/2, \quad q^2 = \overline{u'^2} + \overline{v'^2} + \overline{w'^2} \quad (16)$$

$$\epsilon = \left\langle \frac{\partial u_i}{\partial x_j}, \frac{\partial u_k}{\partial x_j} \right\rangle = b \frac{q^2}{\Lambda} \quad (17)$$

The first two relations are definitions relating turbulent velocity fluctuations to the turbulent kinetic energy; the third equation is an empirical transport law of great importance. Its origin is attributed to Reynolds but no specific reference seems to exist. By the time Batchelor wrote his book(1950), the law was well known to all scientists that studied turbulence. The precise value of the proportionally constant differs in different models and probably the "proportionality constant" is not a true constant but a quantity which is flow-dependent. Since the level of the turbulence and the dominant scale of the fluctuations are parameters of

immediate interest in characterizing a turbulent flow, it is quite natural to develop dynamical equations for these two quantities in terms of the fundamental Navier-Stokes equations themselves. The Reynolds stress closure extends the $k - \epsilon$ idea in the direction of including anisotropy in the stresses. The equation for the two-fold velocity correlation function is ($R_{ij} \equiv \langle u'_i u'_j \rangle$)

$$\frac{\partial}{\partial t} R_{ij} + U_k \frac{\partial}{\partial x_k} R_{ij} + R_{ik} \frac{\partial U_j}{\partial x_k} + \frac{\partial U_i}{\partial x_k} R_{jk} + \quad (18)$$

$$+ \frac{\partial}{\partial x_k} \langle u'_i u'_j u'_k \rangle + \frac{\partial}{\partial x_k} \langle p' u'_i \rangle + \frac{\partial}{\partial x_i} \langle p' u'_j \rangle + \langle p' \left(\frac{\partial u'_i}{\partial x_j} - \frac{\partial u'_j}{\partial x_i} \right) \rangle = \quad (19)$$

$$\nu \nabla^2 R_{ij} - 2\nu \left\langle \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k} \right\rangle \quad (20)$$

The derivations of this equation is carried out in detail in Hinze. Reynolds stress closure brings the ideas of Kolmogoroff on dissipation, Chou on triple velocity correlations and Rotta on the pressure-rate of strain correlation, to bear on the terms in the dynamical equation for the velocity correlation that require closure. With an additional assumption on the behavior of the turbulent scales it is then possible to close the equation for the Reynolds stress. Analogies with the kinetic theory of gases (e.g. with the thirteen moment approximation) are very suggestive although such analogies cannot be based on rigorous statistical reasoning. Reynolds stress models are used very extensively in various aspects of fluid dynamics, in particular in meteorology (Mellor at Princeton).

The frontier of research in spatial turbulent transport lies in efforts made to obtain a closure approximation for the two point velocity correlation tensor. The governing definition and the equations for this tensor are

$$\mathfrak{R}_{ij} = \langle u'_i(x) u'_j(y) \rangle \quad (21)$$

$$\frac{\partial \mathfrak{R}_{ij}}{\partial t} + U_k(x) \frac{\partial \mathfrak{R}_{ij}}{\partial x_k} + U_k(y) \frac{\partial \mathfrak{R}_{ij}}{\partial y_k} + \mathfrak{R}_{ik} \frac{\partial U_j}{\partial y_k} + \frac{\partial U_i}{\partial x_k} \mathfrak{R}_{kj} + \quad (22)$$

$$+ \frac{\partial}{\partial x_k} \langle u'_i(x) u'_j(y) u'_k(y) \rangle + \frac{\partial}{\partial y_k} \langle u'_i(x) u'_j(x) u'_k(y) \rangle + \quad (23)$$

$$+ \frac{\partial}{\partial x_i} \langle p'(x) u'_j(y) \rangle + \frac{\partial}{\partial y_j} \langle p'(y) u'_i(x) \rangle = \nu (\nabla_x^2 + \nabla_y^2) \mathfrak{R}_{ij} \quad (24)$$

A detailed derivation is found in Hinze. Closing this dynamical equation is the simplest systematic closure for the turbulence dynamics because both the intensity and the scale of the turbulence are directly related to the two point correlation tensor. The mathematical relations are:

$$q^2 = \langle u'_i(x) u'_i(x) \rangle = \mathfrak{R}_{ii}(x, x) \quad (25)$$

$$\frac{4q^2}{3} \pi \Lambda \frac{(x+y)}{2} = \int d^3(x'-y') \frac{\mathfrak{R}_{ij}}{(x'-y')^2} \quad (26)$$

$$\mathfrak{R}_{ij}(x, x) = R_{ij}(x) \quad (27)$$

By inspection of these relations, it is clear that the $k - \epsilon$ and the Reynolds stress models are special cases of the two point Reynolds stress model. Furthermore the spectral density is the Fourier transform of the two point velocity correlation so that the spectral model of the two point Reynolds stress closure. Progress in this direction is slow. One reason is that it is difficult to test turbulent transport assumptions for the two point correlations because the tests usually depend on determining the whole flow field (and then comparing with experiments). However, in order to determine the entire flow field, the equations governing the two point correlation tensor must be solved. This task requires solution of the coupled non-linear partial differential equations in 6+1 dimensions. Such equations have not yet found efficient techniques for integration on computers. The contemporary approach of chaos dynamics modeling that is described below offers a very promising alternate line of attack on turbulent flows.

4 Chaos Dynamics Approach

The underlying ideas of modeling turbulent flows via chaos dynamics are best visualised by referring schematically to Lorenz' seminal paper in the Journal of Atmospheric Sciences (Ref. 33). In this work, three major steps are taken to simplify the fluid equations, without distorting the features relevant to onset of instabilities, transition to, and behavior of turbulence. This approach originates with Rayleigh's analysis of a thin layer of fluid heated from below, and his discovery of the parameter relevant for the onset of convective motion (rolls and Benard cells). Rayleigh also correctly determined the critical value of the parameter for the onset of thermal instabilities which generate convection. The first step consists in introducing Fourier analysis of the fluid variables the second step consists in retaining three modes (in the Lorenz formulation of the Benard problem). With his step, the partial differential equations governing the fluid are replaced by a finite set of ordinary differential equations governing the dominant modes of the fluid motion. The third step consists in analyzing the motion of the state point (in the finite dimensional state space obtained from the second step) on a Poincaré surface of section. The section is a cut in the state space on which it is possible to keep track of the successive returns of the state point. Lorenz found that the behavior of the turbulent fluid on the surface of section is governed by a map approximated by the logistic map. We do not expect Lorenz' results for the Benard problem to be completed universal, i.e. applicable to arbitrary flows. It is therefore the major purpose of our effort to find the appropriate analogues for the free atmosphere. The three steps of the Lorenz formalism are shown in the following useful diagram, which essentially outlines very concisely the calculations.

