(2)

REPORT DOCUMENTATION PAGE

AD-A232 916

Form Approved
OMB No. 0704-0188

...to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, ...ving the collection of information. Send comments regarding this burden estimate or any other aspect of this ...n, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson ...e of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
		Final Tech. Rpt. 9/1/88-12/31/89

4. TITLE AND SUBTITLE
The Effect of Symmetry on the Hydrodynamic Stability of and Bifurcation from Planar Shear Flows

5. FUNDING NUMBERS
AFOSR-88-0196
61102F 2304/A4

6. AUTHOR(S)
Thomas J. Bridges

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Worcester Polytechnic Institute
100 Institute Road
Worcester, MA 01609

8. PERFORMING ORGANIZATION REPORT NUMBER
AFOSR-TR- [illegible]

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Air Force Office of Scientific Research
Bolling Air Force Base, D.C. 20332-6448

10. SPONSORING/MONITORING AGENCY REPORT NUMBER
AFOSR-88-0196

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

In this project a new approach to boundary layer transition has been developed based on the use of dynamical systems theory in a spatial setting. The results extend the classic theory of spatial stability into the nonlinear regime and a theory for spatial Hopf bifurcation, spatial Floquet theory, wavelength doubling and spatially quasi-periodic states has been developed and applied to the boundary layer problem. The demonstration of the prevalence of spatially quasi-periodic states (in the Blasius boundary layer) is important for applications because it provides the first mathematically consistent theory for the appearance of spatially quasi-periodic states in shear flows which have been observed in experiments. Exact symmetries in the Navier-Stokes equations and normal form symmetries play a basic role in the theory and require use of equivariant dynamical systems theory. Scenarios for the transition to turbulence are easily postulated in the spatial (convective) framework and a conjecture on the transition to "convective" turbulence through wavelength doubling is introduced.

14. SUBJECT TERMS	15. NUMBER OF PAGES
	68
	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR

91 3 11 124

AFOSR Grant No. 88-0196
Program Manager: Dr. Arje Nachman

Project Final Technical Report
December 1990

The Effect of Symmetry on the Hydrodynamic Stability of and Bifurcation from Planar Shear Flows

THOMAS J. BRIDGES

MATHEMATICAL INSTITUTE
UNIVERSITY OF WARWICK
COVENTRY CV4 7AL

and

DEPT. OF MATHEMATICAL SCIENCES
WORCESTER POLYTECHNIC INSTITUTE
WORCESTER, MASS. 01609

Table of Contents

List of Figures

1. Introduction

Many of the classic fluid flows of great practical importance: boundary layers, channel flows, jets and wakes are *open systems*. That is, they are unbounded in some (or all) spatial directions and therefore do not appear to be natural candidates for application of finite dimensional dynamical systems theory. On the other hand the linear theory of such systems is well understood particularily when the initial instability is *convective* rather than *absolute*. The useful feature of convectively unstable flows is that changes in the flowfield take place in a "boosted" frame of reference $x \mapsto x - ct$ $c \in \mathbb{R}$. The idea of convective instability is intimately linked with the now classic theory of *spatial stability* of shear flows (Gaster [1963]). We introduce a natural generalization (to the nonlinear regime) of the theory of spatial stability (spatial bifurcation theory) and our claim is that *the transition to turbulence in open systems with equilibrium states initially unstable through a convective instability can be analyzed using dynamical systems theory in a spatial setting.*

In the linear theory of spatial stability the frequency ω of a disturbance is treated as real and the (in general complex) wavenumber is the eigenvalue (with $\mathrm{Im}(\alpha) < 0$ corresponding to a *spatially unstable* wave and $\mathrm{Im}(\alpha) > 0$ corresponding to a *spatially stable* wave). In the neutral case the temporal and spatial *linear* theories coincide and a neutral curve can be plotted in three ways as shown in Figure 1.1. Figure 1.1(c) is the relevant neutral curve for spatial bifurcation theory however. In particular, *in spatial bifurcation theory the wavespeed c is treated as a given real parameter*. A sketch of the theory is as follows. Suppose $(U(y), 0)$ is a (parallel) 2D equilibrium state and write the Navier-Stokes equations (in this case the stream function and vorticity variables) perturbed about the equilibrium state *as a spatial evolution equation in the boosted frame $x \mapsto x - ct$*,

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi\\ v\\ \xi\\ \omega\end{pmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0\\ \frac{\partial^2}{\partial y^2} & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & -RU'' & -\frac{\partial^2}{\partial y^2} & R(U-c)\end{bmatrix}\begin{pmatrix}\psi\\ v\\ \xi\\ \omega\end{pmatrix} + \text{nonlinear terms} \qquad (1.1)$$

or succinctly,

$$\frac{\partial}{\partial x}\Phi = \mathbf{L}(c, R) \cdot \Phi + \cdots \tag{1.2}$$

where $v = -\psi_x$, ξ is the vorticity and $\omega = \xi_x$ (see Section 2.1 for details). The idea is to pick any $c \in \mathbf{R}$ (interesting values are those that intersect the neutral curve), increase R and determine all bounded (in the streamwise direction) solutions. In this setting it is straightforward to use the symmetry of the equations and to apply dynamical systems theory to show the existence of a spectacular zoo of spatial structures including spatially quasi-periodic states with 2,3 and possibly 4 independent wavenumbers! The bifurcation sequence begins at the neutral point $R = R_o$ where $\mathbf{L}(c, R_o)\hat{\Phi}(y) = i\alpha_o\hat{\Phi}(y)$ $\alpha_o \in \mathbf{R}$; that is, the linearization of (1.1) has purely imaginary eigenvalues, a two-dimensional center subspace and a point of *spatial Hopf bifurcation*. With no restriction (except for boundedness) placed on the streamwise spatial structure the spatially periodic state will inevitably undergo wavelength doubling (with cascades) and/or secondary bifurcation to spatially quasi-periodic states. In fact a central observation of our work is that spatially quasi-periodic states are prevalent in shear flows (generically occur in the one parameter family of 2D states along the upper branch of the 2D neutral curve in Figure 1.1(c)). Although the theory presented in the sequel is generally applicable to any 2D parallel equilibrium state with a neutral curve as in Figure 1.1 we suppose throughout that the basic equilibrium state is the (parallel) Blasius boundary layer.

Of fundamental importance in transitional shear flows is the origin and subsequent bifurcation of 3D states. However in Section 2 we begin with spatial bifurcations in 2D and show that even in 2D new and interesting spatial structure arises. The 2D Navier-Stokes equations can be written as a spatial evolution equation in a number of ways and two forms are introduced in Sections 2.1 and 2.3 using the stream function & vorticity variables and the primitive variables (which leads to an interesting nonstandard evolution equation) respectively. The basic 2D spatial bifurcation problem is introduced in Section 2.1 and in Section 2.2 a *spatial* secondary "instability" theory is introduced that complements the *temporal* secondary instability theory of Orszag &

Patera [1983] and Herbert [1983,1984]. It is shown how the known structure of wavelength doubling will potentially lead to cascades of wavelength doubling (wavelength "bubbling") and the secondary bifurcation to 2D spatially quasi-periodic states is expected (a demonstation of secondary bifurcation to 2D spatially quasi-periodic states is carried out in Section 5).

Spatial bifurcations with the addition of spanwise structure (three dimensionality) are considered in Section 3. The 3D Navier-Stokes equations are written as an evolution equation in the primitive variables; that is with $\Phi = (v, v_x, w, w_x, p, p_x)^T$ the 3D Navier-Stokes equations can be written as $\frac{\partial}{\partial x}\Phi = \mathbf{L}(c, R)\Phi + \mathbf{N}(\Phi, u; R)$ and u is obtained from the streamwise momentum equations (see Section 3.5 for details). We have not attempted to construct other (spatial) evolution equations for the 3D Navier-Stokes equations but spatial evolution equations for the vorticity & velocity formulation or a vector stream function formulation should also be useful. Any bounded spanwise structure (satisfying the equations) is admissable but with the simple assumption of spanwise periodicity already the number of interesting bifurcations of the streamwise structure is considerable. The assumption of spanwise periodicity leads to an O(2)-equivariance of the evolution equation which is central to the analysis of bifurcating 3D states. In Sections 3.1 to 3.3 O(2)-equivariant (spatial) Hopf bifurcation theory is used to analyze the primary and secondary spatial bifurcation of 3D states that are periodic in both the spanwise and streamwise directions. Section 3.3 contains a generalization of the spatial secondary "instability" theory of Section 2.2 to 3D. Our most useful observation with regard to applications is that all along the upper branch of the 2D neutral curve there exists an interaction between a 2D state with streamwise wavenumber α_1 and a 3D state with streamwise wavenumber α_2 but with *both waves travelling at the same phase speed.* From a theoretical point of view the interaction is a codimension 2 point of an O(2)-equivariant vectorfield with a 6-dimensional center subspace! In Section 4 a formal application of centre-manifold theory and normal form theory is used to show that all along the upper branch of the 2D neutral curve there exists secondary bifurcation to 3D states that are quasi-periodic in the streamwise direction (and periodic in the spanwise direction). The theory is formal simply because the Blasius boundary layer is not an exact solution of the Navier-Stokes equations and the additional neglect of non-parallel terms. For strictly parallel flows with a similar neutral curve (such as plane Poiseuille flow) the theory can be carried through rigorously (Bridges [1991c]), although particular care is always necessary when dealing with the bifurcation of tori.

The theory for the quasi-periodic interaction of a 2D wave with 2 oblique (3D) waves is of great practical interest because it is a mathematically consistent theory for the appearance of quasi-periodic waves in the Blasius boundary layer. Experiments of Kachanov & Levchenko [1984] have shown that a quasi-periodic interaction between a 2D fundamental Tollmien-Schlichting wave with a pair of oblique waves is observed as a robust part of the transition process. The normal form for the quasi-periodic interaction is worked out in Section 4. Some straightforward (although lengthy) calculations are necessary to determine the coefficients in the normal form and this work is in progress (Bridges [1991b]).

In Section 4.2 the interesting idea of *spanwise resonances* is considered briefly. In other words two pairs of oblique waves, one with spanwise wavenumber β and other with spanwise wavenumber $n\beta$ $n = 2, 3, \ldots$ interact. This is a codimension 2 interaction (plot the β and $n\beta$ neutral curves; the point of intersection between the two curves is the interaction point). Such a codimension 2 point occurs for each value of β (β sufficiently small) but the *streamwise* wavenumbers of the two waves will be irrationally related. Although the spanwise resonant interactions occur at Reynold's numbers considerably higher than the 2D-3D interaction of Section 4.1, they are nevertheless of great interest. From a theoretical point of view the interaction corresponds to an 8-dimensional centre-subspace and the normal form indicates the potential for bifurcation to high dimensional tori. From a practical point of view the spanwise resonances introduce new spatial structure that may be important for the transitional boundary layer at higher Reynolds number.

Finally in Section 5 the strictly two-dimensional problem is reconsidered and the "codimension-2 strategy" is used to show secondary bifurcation to 2D spatially quasi-periodic states. The compliant wall problem (Carpenter [1990], Carpenter & Morris [1990]) provides an interesting setting for the analysis because it already contains numerous new parameters. Research of Carpenter & Garrad [1985] has shown that the upper and lower branches of the neutral curve coalesce and detach (see Figure 1.2) when the elastic modulus of the compliant wall (adjacent to the Blasius boundary layer) is reduced. In Section 5 the critical point $E = E_o$ is analyzed and it is shown that at the interaction point of the upper/lower branches the linear Navier-Stokes equations have spatially quasi-periodic solutions with two independent wavenumbers. Application of dynamical systems theory (in particular Section 7.5 of Guckenheimer & Holmes [1983]) shows that the unfolding of the above singularity results in a secondary bifurcation of (spatially) quasi-periodic states (in the nonlinear equations). The singularity in question is not particularly important to the main function of the compliant wall (stabilization and drag reduction) but it nevertheless demonstrates that *secondary bifurcation to spatially quasi-periodic states is to be anticipated even in two-dimensional boundary layers.*

In spite of the fact that our methods are local (centre-manifold theory, normal form theory and local equivariant dynamical systems theory) the existence of quite complex spatial structures in shear flows is demonstrable. On the other hand it is clear that the bifurcations to the various spatial tori and sequences of wavelength doubling will inevitably lead (in some regions of parameter space) to chaotic spatial structure. Will this be related to turbulence? Suppose that a sequence of bifurcations takes place leading to non-trivial spanwise variation and chaotic streamwise structure. The flowfield will indeed be three-dimensional and although the governing equations are "steady" *the streamwise coordinate is in fact* $x - ct$. Therefore a probe at fixed x will register a chaotic flow in time even though time does not appear independently. Assuming there is sufficient three-dimensionality for true turbulence (and an absolute instability does not occur) it is entirely possible that the transition process takes place in the convective frame $x - ct$. We call this structure *convective chaos* or *convective* *turbulence* if indeed it is turbulence. However, in studying the sequence of bifurcations in the convective frame it is important to check for secondary, tertiary, etc. *absolute* instabilities which will force time into the problem independent of $x - ct$.

With the emphasis throughout on spatial bifurcations and spatial invariant manifolds the stability assignments obtained from the normal forms (written as evolution equations in x) will not be applicable. A complete stability analysis of the spatial structures is a very interesting problem and will require the reintroduction of time and consideration of sideband instability (and its generalizations). For spatial states that are periodic in both the streamwise and spanwise direction the spatio-temporal stability can be studied using the theory of sideband instability but the generalization of this concept to study the stability of the spatially quasi-periodic states is by no means clear and will be an interesting area for further research.

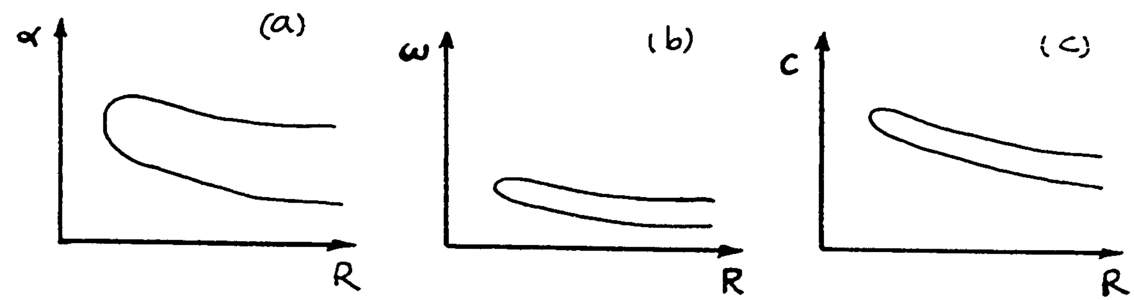

Figure 1.1 Three views of the neutral curve for an example parallel shear flow: (a) temporal stability approach, (b) spatial stability approach and (c) spatial bifurcation approach.

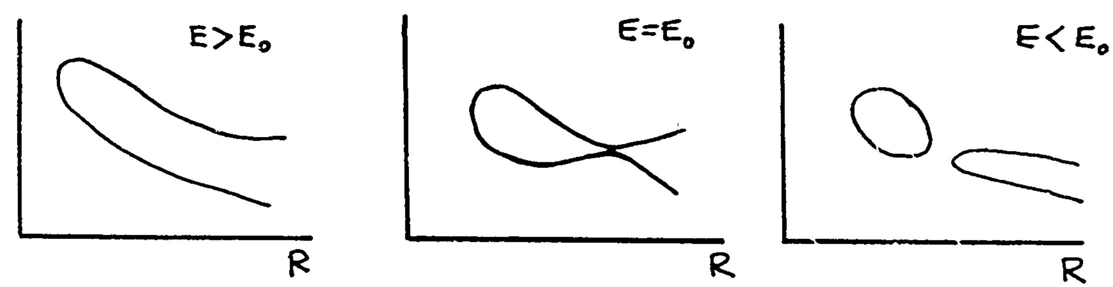

Figure 1.2 Effect of the wall elastic modulus on the neutral curve for the Blasius boundary layer adjacent to a compliant wall (after Carpenter & Garrad [1985, Figure 11]).

2. Spatial bifurcations in two dimensions

The 2D Navier-Stokes equations are considered in a steady-frame ($x \mapsto x - ct$) *with c given* and we look for all possible states moving at speed c but with *various spatial structure* that bifurcate from the equilibrium state. For definiteness suppose that the basic equilibrium state is the Blasius boundary layer (U, V) with $U(x,y) = f'(\eta)$, $V(x,y) = \frac{1}{2}(xR)^{-1/2}(\eta f'(\eta) - f)$, $\eta = y(R/x)^{1/2}$ and $f(\eta)$ satisfying the Blasius equation $2f''' + ff'' = 0$ on $\eta \in [0,\infty)$. The Blasius solution is a troublesome equilibrium state in that it satisfies the Navier-Stokes equations only asymptotically and a proper existence theory and stability theory requires careful use of asymptotic and multiple deck theory (Smith [1979]). We assume here that $x = \mathcal{O}(R)$ and invoke the parallel flow approximation. Then $U = \mathcal{O}(1)$, $V = \mathcal{O}(R^{-1})$ and the two-dimensional Navier-Stokes equations perturbed about the (parallel) Blasius boundary layer solution are

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &(U - c)\frac{\partial u}{\partial x} + U_y v + \frac{\partial p}{\partial x} - \frac{1}{R}\Delta u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0 \\ &(U - c)\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} - \frac{1}{R}\Delta v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0. \end{aligned}\right\} \tag{2.1}$$

In the case where the equilibrium state is strictly parallel the set (2.1) is exact. In the set of equations (2.1) time has been eliminated by the shift $x \mapsto x - ct$; in particular we are looking for *steady* bifurcations. Time can be reintroduced for a stability analysis or if there is a bifurcation to a non-trivial temporal state. The crucial difference in (2.1) from the usual theory is that *c is treated as a given real number*. The idea is to treat the set (2.1) as an evolution equation in x; that is, a pde with y as the "spatial" variable and x as the "time-like" variable. With parameters (c, R) we look for all the usual bifurcations in evolution equations: Hopf-bifurcation, period-doubling and torus bifurcation, etc, except that in the spatial setting these bifurcations will correspond to a *spatial*-Hopf bifurcation, *wavelength* doubling and *spatially* quasiperiodic states.

The basic bifurcation from the Blasius boundary layer is determined by studying the spectrum of the linearization of (2.1),

$$\left.\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\\ &(U-c)\frac{\partial u}{\partial x}+U_y v+\frac{\partial p}{\partial x}-\frac{1}{R}\Delta u=0\\ &(U-c)\frac{\partial v}{\partial x}+\frac{\partial p}{\partial y}-\frac{1}{R}\Delta v=0.\end{aligned}\right\} \tag{2.2}$$

Now let $(u,v,p)=e^{\lambda x}(\hat{u},\hat{v},\hat{p})$ then with $\hat{p}_{yy}+\lambda^2\hat{p}=-2\lambda U_y\hat{v}$ the set (2.2) can be reduced to a modified (real) form of the familiar Orr-Sommerfeld equation

$$\mathbf{L}\cdot\hat{v}=\left(\frac{d^2}{dy^2}+\lambda^2\right)^2\hat{v}+\lambda RU_{yy}\hat{v}-R\lambda(U-c)\left(\frac{d^2}{dy^2}+\lambda^2\right)\hat{v}=0 \tag{2.3}$$

We have purposefully used $e^{\lambda x}$ rather than the usual $e^{i\alpha x}$ (which we'll revert back to shortly) to make an analogy with the dynamical systems approach.

The Orr-Sommerfeld equation is discretized using a finite expansion of $\hat{v}(y)$ (with $[0,\infty)$ mapped to $[-1,+1]$ using an algebraic transformation) in a Chebyshev series reducing (2.3) to an algebraic nonlinear in the parameter eigenvalue problem (see Bridges & Morris [1987]) for this type of reduction). The differential eigenvalue problem (2.3) is then reduced to the algebraic eigenvalue problem

$$\mathbf{D}_4(\lambda)\{\hat{v}\}=[\mathbf{C}_0\lambda^4+\mathbf{C}_1\lambda^3+\mathbf{C}_2\lambda^2+\mathbf{C}_3\lambda+\mathbf{C}_4]\{\hat{v}\}=0. \tag{2.4}$$

Note that in the fixed frequency spatial stability problem (Bridges & Morris [1987]) the matrices $\mathbf{C}_0,\ldots,\mathbf{C}_4$ are *complex* but with fixed wavespeed $c\in\mathbf{R}$ the matrices $\mathbf{C}_0,\ldots,\mathbf{C}_4$ are *real*. The eigenvalue will in general be complex, but with real matrices the numerical computation can be carried out with greater efficiency. The eigenvalue problem (2.4) is solved using the methods of Bridges & Morris [1984] for nonlinear in the parameter eigenvalue problems.

Given (c,R) the eigenvalue problem (2.4) produces a spectrum of spatial eigenvalues. The idea in spatial bifurcation theory is to look for *bounded* solutions of (2.3); that is, there are admissible spatial states bifurcating from the Blasius boundary layer if and only if there exists an eigenvalue λ of (2.4) with $\mathrm{Re}(\lambda)=0$. Eigenvalues with $\mathrm{Re}(\lambda)\neq 0$

are not admissible as bifurcation points for spatial states because they are unbounded as $x \to +\infty$ or $x \to -\infty$. Note that we are not concerned with eigenvalues that lie off the imaginary axis (whether in the right or left half plane). Ultimately stability is determined by checking the initial value problem (reintroducing time!). If $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\lambda) = 0$ there is (potentially) a bifurcation to a new equilibrium state (although this is not expected to occur for the Blasius boundary layer) whereas if $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\lambda) \neq 0$ there is a (spatial)-Hopf bifurcation (assuming the usual non-degeneracy conditions on Hopf bifurcation) from the equilibrium state to a spatially periodic state.

Solving $|\mathbf{D}_4(\lambda)| = 0$ and $\mathrm{Re}(\lambda) = 0$ results in the well known neutral curve for the Blasius boundary layer shown in Figure 2.1 (although it is usually plotted in (ω, R) or (α, R) space). In particular there is at most one pair of eigenvalues on the $\mathrm{Im}(\lambda)$ axis. Figure 2.3 shows an example of the spectrum of the eigenvalue problem (2.4) (there is also a (stable) continuous spectrum of (2.3) (Grosch & Salwen [1978]) that will appear as discrete in the finite-dimensional approximation). Accurate calculations of the (c, R) curve with the associated value of α are given in Table 2.1. Note that there is a finite interval in wavespeed $c \in (c_1, c_2)$ in which the neutral curve exists where $c_1 \approx .22$ and $c_2 \approx .401$. We call c admissible if $c \in (c_1, c_2)$. It is clear from Figure 2.2 that if $c \in (c_1, c_2)$ and fixed and R is increased, a spatial-Hopf bifurcation occurs as R crosses the neutral curve. There is a continuum of Hopf bifurcation points (varying c) and this will have consequences with regard to stability (the possibility of sideband instability will have to be considered) but nevertheless, for fixed $c \in (c_1, c_2)$ a classic Hopf bifurcation takes place as R crosses the neutral curve. We call it a spatial Hopf bifurcation because the "frequency" associated with the bifurcation is in fact the streamwise *wavenumber*.

2.1 Primary spatial Hopf bifurcation

The set of equations (2.1) can be written as an evolution in x in the following way. Introduce the stream function ψ with $u = \psi_y$, $v = -\psi_x$ and the vorticity $\xi = -\Delta\psi$.

Then the vorticity and stream function equation set can be written as

$$\frac{\partial}{\partial x}\Phi = \mathbf{L}(c,R)\cdot\Phi + \mathbf{N}(\Phi;R) \tag{2.5}$$

where

$$\Phi = \begin{pmatrix}\psi\\ v\\ \xi\\ \omega\end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix}\psi\\ -\psi_x\\ \xi\\ \xi_x\end{pmatrix}, \quad \mathbf{N}(\Phi,R) = R\begin{pmatrix}0\\ 0\\ 0\\ \omega\psi_y + v\xi_y\end{pmatrix} \tag{2.6a}$$

and

$$\mathbf{L}(c,R) = \begin{pmatrix}0 & -1 & 0 & 0\\ \frac{\partial^2}{\partial y^2} & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & -RU'' & -\frac{\partial^2}{\partial y^2} & R(U-c)\end{pmatrix}. \tag{2.6b}$$

Taking $\Phi = e^{\lambda x}\hat{\Phi}$ results in the eigenvalue problem $\mathbf{L}(c,r)\hat{\Phi} = \lambda\hat{\Phi}$ which reduces to the Orr-Sommerfeld problem (2.3) for the stream function perturbation. Suppose $c \in (c_1, c_2)$ and $R = R_o$ is a point on the neutral curve. Then $\mathrm{Re}(\lambda) = 0$, $\mathrm{Im}(\lambda) = \alpha_o$ and there exists an eigenfunction

$$\mathbf{L}(c,R)\hat{\Phi} = i\alpha_o\hat{\Phi}. \tag{2.7}$$

It is now straightforward to apply the Hopf bifurcation theorem using a formal centre-manifold reduction (Coullet-Spiegel [1983]). Scale $x \mapsto \alpha x$ so that the wavenumber appears in equation (2.5): $\alpha\Psi_x = \mathbf{L}(c,R)\cdot\Psi + \mathbf{N}(\Psi,R)$. Write any solution of (2.5) as

$$\Phi(x,y) = A(x)\hat{\Phi}(y) + \overline{A(x)\hat{\Phi}(y)} + \Psi(x,y)$$

then the scaled version of equation (2.5) is transformed to

$$\alpha\frac{dA}{dx} = f_1(A,\overline{A},\Psi) \tag{2.8a}$$

$$\alpha\frac{d\overline{A}}{dx} = \overline{f_1(A,\overline{A},\Psi)} \tag{2.8a}$$

$$\alpha\frac{d\Psi}{dx} = f_2(A,\overline{A},\Psi). \tag{2.8c}$$

At least locally (2.8c) can be solved for Ψ as a function of A and $\overline{A}$ (Coullet & Spiegel [1983]). Then back substitution of Ψ into (2.8a) results in a vectorfield on C. Application of normal form theory (see Guckenheimer & Holmes [1983, p.142]) results in the (formal) normal form for the spatial-Hopf bifurcation,

$$\frac{dA}{dx} = A\cdot F(R-R_o, \alpha-\alpha_o, |A|^2). \tag{2.9}$$

The reduction of (2.5) to (2.9) is formal simply because the Blasius solution is not a truly parallel solution of the Navier-Stokes equations and we have neglected the non-parallel terms. However the basic idea of a centre-manifold reduction for a spatial bifurcation can be rigorously justified (Vanderbauwhede & Iooss [1990]) when an exact equilibrium state is used such as plane Poiseuille flow (Bridges [1991c]). The actual numerical calculations required for the normal form (2.9) will be discussed in Section 3.2 and are contained in Bridges [1991b] as a special case of the 3D calculations.

Expand F in a Taylor series,

$$F = F_R^o(R - R_o) + F_\alpha^o(\alpha - \alpha_o) + F_{A\bar{A}}^o|A|^2 + \cdots . \tag{2.10}$$

The imaginary part of (2.10) is solved for $(\alpha - \alpha_o)$ and back substitution into the real part of (2.10) results in the usual pitchfork bifurcation

$$\frac{da}{dx} = a\,g(R - R_o, a^2), \qquad g = g_R^o(R - R_o) + g_{a^2}^o a^2 + \cdots .$$

Supposing $g_R^o > 0$, a supercritical pitchfork bifurcation occurs if $g_{a^2} < 0$ and a sub-critical pitchfork bifurcation occurs if $g_{a^2} > 0$. Stability of the bifurcating states does not follow directly from (2.9) when the bifurcation is supercritical. For a satisfactory stability analysis time must be reintroduced and sideband instability considered.

From the physical point of view the normal form provides two crucial pieces of information: (a) the direction in parameter space (Reynold's number in this case) in which the nonlinear spatially periodic states persist ($R < R_o$ or $R > R_o$) and (b) how the *wavenumber* changes along a branch of periodic states. We are particularily interested in whether the *wavenumber goes to zero* along the branch leading to spatial homoclinic bifurcation.

2.2 Secondary bifurcations and spatial Floquet theory

One of the central features of the fixed-wavespeed spatial-bifurcation theory is that it is clear how more complex *spatial* bifurcations can arise (and indeed are expected).

In Section 2.1 we showed that for for fixed $c \in (c_1, c_2)$, a spatial-Hopf bifurcation leads to a branch of spatially periodic solutions which is entirely analogous to a branch of periodic orbits in a finite-dimensional dynamical system. If we follow a branch of periodic solutions then we can expect to encounter period-doubling and/or secondary bifurcation to a quasi-periodic flow. The idea here is to use Floquet theory in space; that is, to study the *spatial* Floquet multipliers along the branch of *spatially* periodic solutions. Let $(u, v, p) \mapsto (u+\xi, v+\eta, p+q)$ with (u, v, p) a periodic solution satisfying (2.1). In the usual way substitute $(u+\xi, v+\eta, p+q)$ into the Navier-Stokes equations and linearize about the branch of periodic solutions. The result is the following system with periodic coefficients

$$\left.\begin{aligned}
&\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} = 0 \\
&(u + U - c)\frac{\partial \xi}{\partial x} + u_x \xi + (U_y + u_y)\eta + v\frac{\partial \xi}{\partial y} + \frac{\partial q}{\partial x} - \frac{1}{R}\Delta \xi = 0 \\
&(u + U - c)\frac{\partial \eta}{\partial x} + v_x \xi + v_y \eta + v\frac{\partial \eta}{\partial y} + \frac{\partial q}{\partial y} - \frac{1}{R}\Delta \eta = 0.
\end{aligned}\right\} \tag{2.11}$$

This set can be simplified by introducing a perturbation stream function; let $\xi = \partial\phi/\partial y$ and $\eta = -\partial\phi/\partial x$ then the second and third of (2.11) can be combined into

$$\begin{aligned}
&\Delta\Delta\phi - R(u + U - c)\frac{\partial}{\partial x}\Delta\phi - Rv\frac{\partial}{\partial y}\Delta\phi + R(v_{xx} - u_{xy})\frac{\partial\phi}{\partial y} \\
&\qquad + R(U_{yy} + y_{yy} - v_{xy})\frac{\partial\phi}{\partial x} = 0.
\end{aligned} \tag{2.12}$$

Now $\phi(x, y)$ is not necessarily periodic in x but by Floquet's theorem

$$\phi(x, y) = e^{\gamma x}\hat{\phi}(x, y) \qquad \gamma \in \mathbb{C}$$

and $\hat{\phi}(x, y)$ *is* periodic in x of the same period as the basic state (u, v, p). Substitution of $\phi = e^{\gamma x}\hat{\phi}$ into (2.6) results in the following eigenvalue problem for the Floquet exponent γ,

$$\mathbf{L} \cdot \hat{\phi} = \gamma^4 \mathbf{L}_0\hat{\phi} + \gamma^3 \mathbf{L}_1\hat{\phi} + \gamma^2 \mathbf{L}_2\hat{\phi} + \gamma \mathbf{L}_3\hat{\phi} + \gamma \mathbf{L}_4\hat{\phi} = 0 \tag{2.13}$$

where

$$\mathbf{L}_0\hat{\phi} = \hat{\phi}$$

$$\mathbf{L}_1\hat{\phi} = 4\frac{\partial\hat{\phi}}{\partial x} - R(u + U - c)\hat{\phi}$$

$$\mathbf{L}_2\hat{\phi} = 2\hat{\Delta}\hat{\phi} + 4\frac{\partial^2\hat{\phi}}{\partial x^2} - 3R(u + U - c)\frac{\partial\hat{\phi}}{\partial x} - Rv\frac{\partial\hat{\phi}}{\partial y}$$

$$\mathbf{L}_3\hat{\phi} = 4\hat{\Delta}\frac{\partial\hat{\phi}}{\partial x} - R(u + U - c)\left(\hat{\Delta}\hat{\phi} + 2\frac{\partial^2\hat{\phi}}{\partial x^2}\right) - 2Rv\frac{\partial^2\hat{\phi}}{\partial x\partial y} + R(U_{yy} + u_{yy} - v_{xy})\hat{\phi}$$

$$\mathbf{L}_4\hat{\phi} = \hat{\Delta}\hat{\Delta}\hat{\phi} - R(u + U - c)\hat{\Delta}\frac{\partial\hat{\phi}}{\partial x} - Rv\hat{\Delta}\frac{\partial\hat{\phi}}{\partial y} + R(U_{yy} + u_{yy} - v_{xy})\frac{\partial\hat{\phi}}{\partial x}.$$

Note that the eigenvalue problem (2.13) is a nonlinear in the parameter eigenvalue problem for the Floquet exponent γ. It is reminiscent of the classic spatial stability eigenvalue problem; in fact it is the *spatial* form of the secondary "instability" problem.

The above theory for secondary bifurcation using Floquet theory is similar to Herbert's [1983,1984] theory for subharmonic bifurcation but there is a subtle difference. The eigenvalue problem (2.13) is the *spatial* secondary "instability" problem whereas Herbert's theory is a *temporal* secondary instability theory. In Herbert's theory the spatial Floquet exponent is treated as fixed (generally so that the spatial multiplier lies at -1) and the temporal exponent is solved for. *In (2.13) the temporal exponent is absent since we are looking for secondary steady states; that is, states that move at the given speed c but have more complex spatial structure.*

Given a Fourier-Chebyshev representation for the basic spatially periodic state, (u, v, p), the eigenvalue problem (2.13) can be discretized by expanding $\hat{\phi}$ in a finite Fourier-Chebyshev series. The result is an algebraic nonlinear in the parameter eigenvalue problem

$$[\mathbf{D}_4(\gamma)] \cdot \{\hat{\phi}\} = [\mathbf{C}_0\gamma^4 + \mathbf{C}_1\gamma^3 + \mathbf{C}_2\gamma^2 + \mathbf{C}_3\gamma + \mathbf{C}_4] \cdot \{\hat{\phi}\} = 0 \qquad (2.14)$$

with the Floquet exponents obtained as roots of $|\mathbf{D}_4(\gamma)| = 0$. Numerical methods for nonlinear eigenvalue problems of the type (2.14) can be found in Bridges & Morris [1984] and Pearlstein & Goussis [1988] and references therein. Given the spectrum of exponents, the Floquet multipliers are given by $\exp(2\pi\gamma/\alpha)$. Suppose that the branch of spatially periodic solutions is parametrized by a parameter ϵ. Then two interesting possibilities as ϵ is varied are shown in Figure 2.3. We are not concerned with where the bulk of the spectrum of the eigenvalue problem lies in the complex plane (although most of the multipliers will probably lie in the interior of the unit circle); any multiplier not on the unit circle is inadmissable as a bounded spatial state. Therefore we vary ϵ until the multiplier lies on the unit circle. If the multiplier passes through -1 we expect a wavelength doubling bifurcation and if complex-conjugate multipliers pass through the unit circle at points other than ± 1 we expect a bifurcation to a spatially quasiperiodic state.

Numerical calculations are necessary to obtain precisely where secondary bifurcations occur. However, spatially quasi-periodic states are to be expected. One way to show this is to introduce a singularity (in the neutral curve) which results in a larger dimensional centre-manifold, then look for complex dynamics in the unfolding. In fact, in Section 5 it is shown that the compliant wall problem has a singularity of this form from which we can show the existence of secondary branches of spatially quasi-periodic states (and possibly spatially quasi-periodic states with *three* frequencies!).

The wavelength doubled solution can again double its wavelength with a possible cascade to spatially chaotic states (not turbulent though since we are restricted to two-dimensions but the 3D problem is considered in Section 3). An interesting theory for wavelength doubling cascades can be constructed as follows. Note that from Figure 2.1 that for fixed $c \in (c_1, c_2)$, there is only a finite band in R in which an unstable region occurs. From numerical calculations of Herbert [1975] we expect the nonlinear branch of periodic solutions to form a global loop as shown in Figure 2.4. The global wavelength doubling structure of loops can be modelled by a one-dimensional *two-parameter* map. For example consider

$$x_{n+1} = x_n\,(1 - (R - R_o)^2 + \gamma - x_n^2). \tag{2.15}$$

The fixed points of the map (2.15) are as shown in Figure 2.4(a). Iterating the map while increasing γ (corresponds to decreasing the wavespeed c) results in secondary, tertiary, etc loops of period doubled points (Bridges [1991e]). Figure 2.5 shows a finite cascade of wavelength doubling. If γ is increased further the cascade will become infinite resulting in chaos in a thin subinterval of R. It is very likely that this is the structure of wavelength doublings in shear flows. Secondary period-doubling bifurcations from the subcritical loop in Figure 2.4(b) can be modelled in a similar fashion using the one-dimensional map

$$x_{n+1} = x_n(1 - (R - R_o)^2 + \gamma - (\beta + 2m(R - R_o))x_n^2 - x_n^4) \tag{2.16}$$

where $\beta, m \in \mathbf{R}$, $0 < m < 1$, $\gamma > 0$ and $\beta < -\sqrt{4m^2\gamma}$. The period doubling structure for the map (2.16) is shown in Figure 2.6 with a finite cascade in Figure 2.6(a) and an infinite cascade leading to a region of chaos in Figure 2.6(b). Note that although the subcritical and supercritical loops in Figure 2.4(a) and (b) differ significantly, the secondary structure and cascade structure is about the same in both cases.

Note that the above theory is restricted to two spatial dimensions. If indeed spatial chaos occurs it will not be turbulence. The role of three-dimensionality is considered in Section 3. On the other hand, three-dimensionality does not affect the cascade theory shown in Figure 2.5, it simply results in non-trivial spanwise structure along with the streamwise cascade.

2.3 Spatial evolution of the primitive variables

As an alternate to the evolution equation (2.5) where the stream function vorticity variables are used, an evolution equation for the primitive variables (u, v, p) can be constructed although it has a nonstandard form (involving an evolution equation and a constraint). The idea is to evolve (in x) the *pressure* and *vertical velocity* and determine the streamwise velocity using a constraint (a differential equation without x-derivatives). First construct a Poisson equation for the pressure by taking the divergence of the momentum equations in (2.1),

$$\Delta p = -2\left(U_y \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right). \tag{2.17}$$

Now let

$$\Phi = \begin{pmatrix} v \\ V \\ p \\ q \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} v \\ v_x \\ p \\ p_x \end{pmatrix} \tag{2.18}$$

Then the Poisson equation (2.17) and the vertical momentum equation can together be written as an evolution equation in x,

$$\frac{\partial}{\partial x}\Phi = \mathbf{L}(c, R) \cdot \Phi + \mathbf{N}(\Phi, u; R) \tag{2.19}$$

where

$$\mathbf{L}(c, R) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\partial^2}{\partial y^2} & R(U-c) & R\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2U_y & -\frac{\partial^2}{\partial y^2} & 0 \end{pmatrix} \text{ and } \mathbf{N} = \begin{pmatrix} 0 \\ R(uV + v\frac{\partial v}{\partial y}) \\ 0 \\ -2(\frac{\partial v}{\partial y})^2 - 2V\frac{\partial u}{\partial y} \end{pmatrix}. \tag{2.20}$$

Note however that u appears in the nonlinear term but is not a component of Φ. However, the streamwise momentum equation can be written as a differential equation in y alone,

$$q + U_y v - (U-c)v_y + \frac{1}{R}(V_y - u_{yy}) - uv_y + vu_y = 0, \tag{2.21}$$

and at each value of x, u is obtained from (2.21) for use in (2.20).

The evolution equation for (u, v, p) is non-standard in that it involves evolution of (v, p) with a *constraint* to determine the streamwise velocity. Nevertheless it is a useful form of the equations; in particular, the framework (2.19)-(2.21) is easily extended to the 3-dimensional Navier-Stokes equations (this is carried out in Section 3.5) and the usual centre-manifold and bifurcation theory is still applicable.

Table 2.1

Coordinates in the (c, R) plane and wavenumber along the neutral curve for the Blasius boundary layer

Upper branch			Lower branch		
R	c	α_o	R	c	α_o
4000	.2956	.2689	4000	.22875	.0918
3500	.3028	.2777	3500	.2368	.0967
3000	.31065	.2881	3000	.2461	.1028
2600	.3185	.2976	2560	.2552	.1097
2500	.3206	.3001	2500	.2568	.1106
2400	.3229	.3029	2360	.2608	.1135
2200	.3276	.3086	2160	.26655	.1178
2000	.3329	.3148	2000	.27105	.12175
1800	.3387	.3215	1960	.2726	.1229
1600	.3451	.3286	1760	.2798	.1290
1500	.3489	.3322	1560	.2881	.1364
1400	.3528	.3362	1500	.29085	.1389
1200	.3616	.3440	1360	.2981	.1456
1000	.3718	.3515	1260	.3034	.1513
900	.3777	.3545	1160	.3105	.1578
800	.3843	.3562	1060	.3178	.1655
750	.3877	.3562	1000	.3223	.17095
730	.38915	.3559	960	.3259	.1749
660	.3941.	.3532	900	.3313	.1814
650	.3947	.3524	850	.3363	.1879
600	.3981	.3469	800	.34205	.1950
580	.3992	.3432	750	.3484	.2035
575	.3993	.3421	700	.3550	.2135
560	.4001	.3379	650	.3630	.2259
550	.4004	.3344	600	.37205	.2419
540	.4002	.3293	580	.37615	.2499
530	.4002	.3241	560	.3812	.2597
525	.3995	.3184	550	.38365	.2656
520	.3951	.2974	540	.3864	.2725
			530	.3987	.2815
			525	.3945	.2857

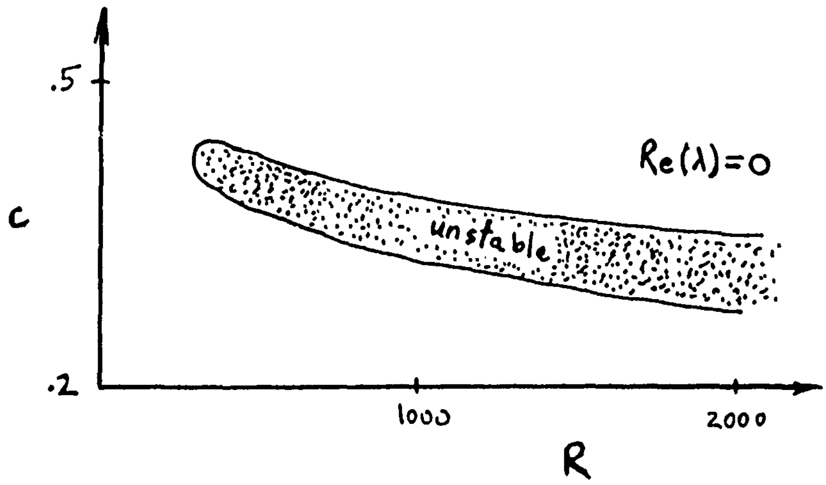

Figure 2.1 Neutral curve in the (c, R) plane for the (parallel) Blasius boundary layer.

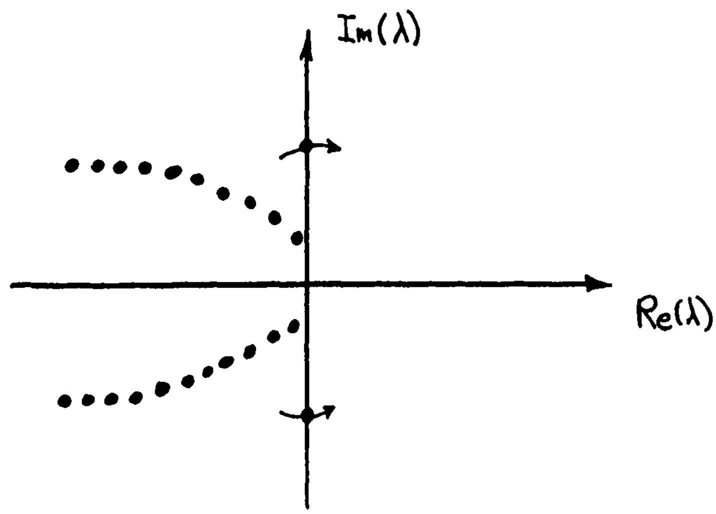

Figure 2.2 Spectrum of the modified (real) Orr-Sommerfeld equation for fixed $c \in (c_1, c_2)$ as R intersects the neutral curve.

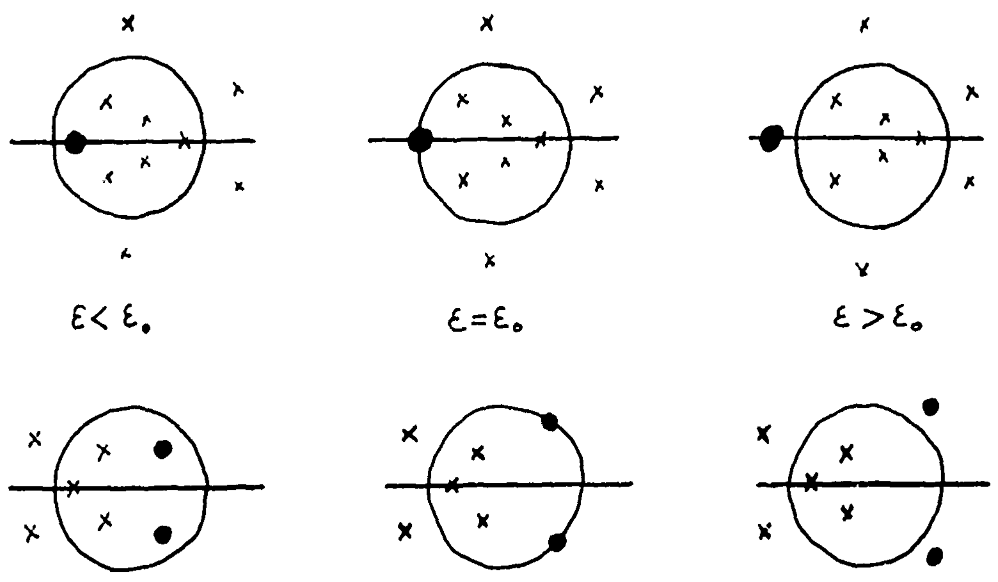

Figure 2.3 Possible movement of the spatial Floquet multipliers $\exp(2\pi\gamma/a)$ as a function of ϵ.

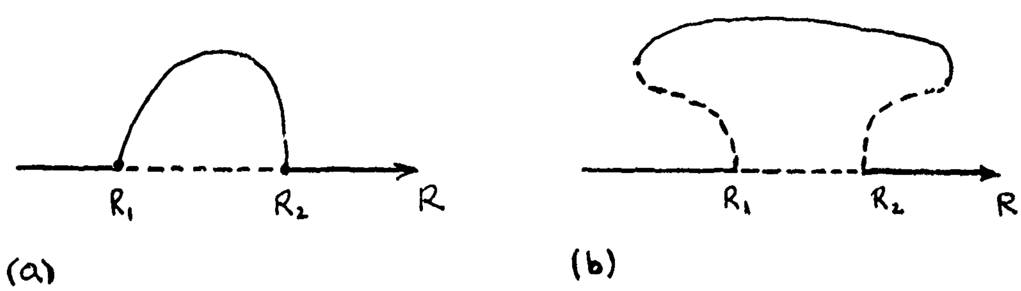

Figure 2.4 Global loop structure for fixed c of spatially periodic states bifurcating from the Blasius boundary layer: (a) supercritical loop and (b) subcritical loop.

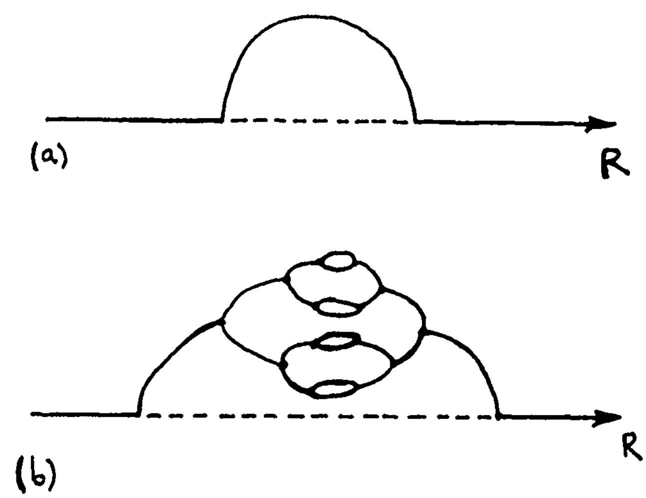

Figure 2.5 Global period-doubling loops with a finite cascade in the map (2.15): (a) $\gamma < 1$ resulting in an absence of period-doubling and (b) $\gamma > 1$ (in particular $\gamma = 1.30$) resulting in three period-doubing bifurcations.

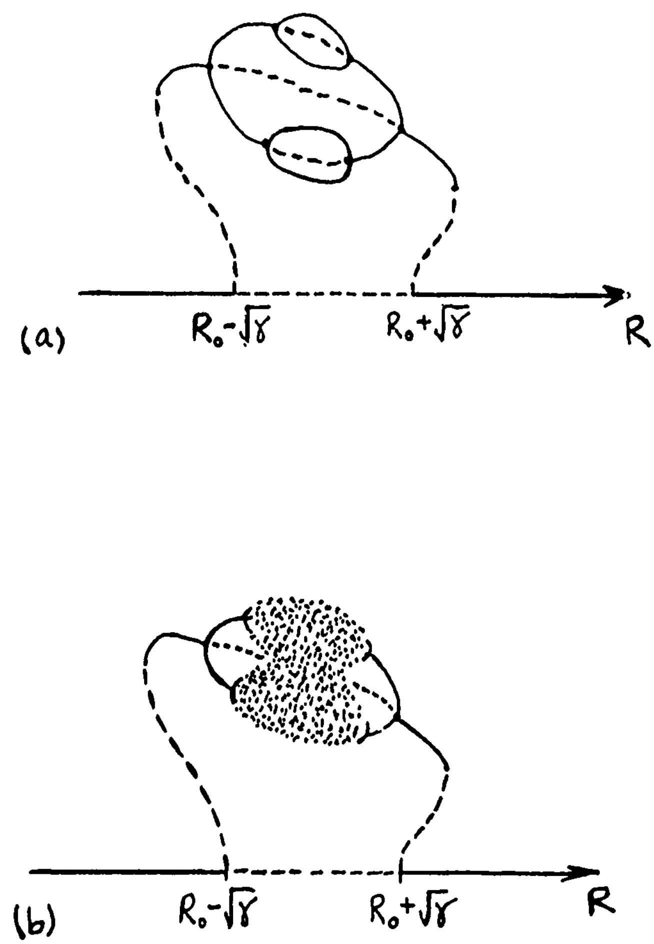

Figure 2.6 Finite and infinite period-doubling cascade in the map (2.16) where the primary loop is subcritical with $m = \frac{1}{2}$, $\beta = -\sqrt{\gamma} - .2$ and (a) $\gamma = .21$ and (b) $\gamma = .24$.

3. Spatial bifurcations in three dimensions

In this section we continue to treat the steady-state (in a moving frame) Navier-Stokes equations as an evolution equation in the streamwise coordinate but with the addition of spanwise variation (three dimensionality). The basic equilibrium state is taken to be the (parallel) Blasius boundary layer although the theory is generally applicable to any two-dimensional parallel equilibrium state. With the shift $x \mapsto x - ct$ the generalization of the equation set (2.1) to three dimensions is

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\ &(U-c)\frac{\partial u}{\partial x} + U_y v + \frac{\partial p}{\partial x} - \frac{1}{R}\Delta u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0 \\ &(U-c)\frac{\partial v}{\partial x} \qquad\quad + \frac{\partial p}{\partial y} - \frac{1}{R}\Delta v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = 0 \\ &(U-c)\frac{\partial w}{\partial x} \qquad\quad + \frac{\partial p}{\partial z} - \frac{1}{R}\Delta w + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = 0 \end{aligned}\right\} \tag{3.1}$$

The addition of z (the spanwise coordinate) introduces nontrivial symmetry that will be fundamental to the analysis of three-dimensional states; in particular, a translation and reflection in z. The translation and reflection in z follow from the fact that the basic state is two-dimensional. The z-reflection generates the group $\mathbf{Z}_2^\kappa = \langle\kappa\rangle$ with action on functions given by $\kappa \cdot f(x,y,z) = f(x,y,-z)$. The set (3.1) is $\mathbf{Z}_2^\kappa$-equivariant and $(u(x,y,-z), v(x,y,-z), -w(x,y,-z), p(x,y,-z))$ is a solution of (3.1) whenever $(u(x,y,z), v(x,y,z), w(x,y,z), p(x,y,z))$ is a solution; that is, there is a $\mathbf{Z}_2^\kappa$-orbit of solutions. Note that the existence of a $\mathbf{Z}_2^\kappa$-orbit of solutions is true regardless of solution type; even chaotic trajectories have a $\mathbf{Z}_2^\kappa$-orbit which has important consequences with regard to division and multiplicity of attracting sets.

The translation invariance in z of the set (3.1) results in a group orbit of solutions as well (an arbitrary translate in z of a solution is also a solution). However if (u,v,w,p) is taken to be *periodic* in z (an assumption; more complex spanwise spatial structure is possible and this is considered in Section 3.4) then the translation group is reduced to the *compact* group SO(2). Combining SO(2) with $\mathbf{Z}_2^\kappa$ results in an O(2)-equivariance of the equation set (3.1). In our subsequent analysis the O(2)-equivariance of the set (3.1) is the basic organizing feature of the three-dimensional states.

In three dimensions the basic spatial-Hopf bifurcation introduced in Section 2.1 persists but the O(2)-equivariance results in a higher dimensional (spatial) centre-manifold, a higher multiplicity of spatially periodic states and the potential for more complex "dynamics" (i.e. more complex spatial bifurcations). For the spatial Hopf bifurcation with O(2) symmetry we adapt the well-developed theory of O(2)-equivariant Hopf bifurcation (Golubitsky & Roberts [1987], Golubitsky, Stewart & Schaeffer [1988]). The O(2)-equivariant spatial-Hopf bifurcation is a *primary* bifurcation to 3D states and is treated in Section 3.1.

In Section 3.3 we introduce a "spatial" secondary instability theory where a primary spatially periodic two-dimensional state bifurcates to a three-dimensional state *at finite amplitude in a steady frame*. This is to be compared with the *temporal* secondary instability theory due to Orszag & Patera [1983] and Herbert [1983,1984]. The 2D → 3D spatial secondary "instability" theory is similar to the theory introduced in Section 2.2 but includes nontrivial (periodic) spanwise variation. Linearization about the 2D state results in a system with periodic (in x) coefficients to which spatial Floquet theory is applied. The system will differ from the system (2.11). The spatial Floquet exponents now depend on *two parameters*: the parametrized branch and β (the spanwise wavenumber of the perturbation). Our approach differs from the temporal theories of Orszag & Patera and Herbert in that the temporal exponent is set to zero; that is, we look for bifurcations to *bounded* steady-states with more complex spatial structure in x (wavelength doubled, tripled, etc. or quasi-periodic) but periodic in z.

It is clear that if a primary state exists that is periodic in both the streamwise direction *and* the spanwise direction (as obtained in Section 3.1 or as a secondary bifurcation as in Section 3.3), it is possible to have secondary bifurcations in *both* the streamwise *and* spanwise directions. In particular, it is possible to have a bifurcation to states with more complex *spanwise* spatial structure. The theory for spatial bifurcation in (x, z) will involve spatial Floquet theory in two space dimensions and is discussed in Section 3.4.

3.1 O(2)-equivariant spatial Hopf bifurcation

To determine bifurcation points from the equilibrium state, the set (3.1) is linearized resulting in

$$\left.\begin{aligned} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}&=0 \\ \left[(U-c)\frac{\partial}{\partial x}-\frac{1}{R}\Delta\right]\begin{pmatrix} u \\ v \\ w \end{pmatrix}+\nabla p+U_y v\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}&=0 \end{aligned}\right\} \tag{3.2}$$

which can be reduced to the two decoupled systems

$$\left.\begin{aligned} \Delta v-R(U-c)\frac{\partial v}{\partial x}-R\frac{\partial p}{\partial y}&=0 \\ \Delta p+2U_y\frac{\partial v}{\partial x}&=0 \end{aligned}\right\} \tag{3.3}$$

and

$$\left.\begin{aligned} \Delta u-R(U-c)\frac{\partial u}{\partial x}&=R\frac{\partial p}{\partial x}+RU_y v \\ \Delta w-R(U-c)\frac{\partial w}{\partial x}&=R\frac{\partial p}{\partial z}. \end{aligned}\right\} \tag{3.4}$$

Taking $u(x,y,z)=e^{\lambda x}[u_1(y)\cos\beta z+u_2(y)\sin\beta z]$ the secondary eigenvalue problem (3.4) reduces to

$$\frac{\partial^2 u_j}{\partial y^2}+[\lambda^2-\beta^2-\lambda R(U-c)]u_j=0 \qquad j=1,2 \tag{3.5}$$

with appropriate boundary conditions. It is easy to show that if $c\in\mathbb{R}$ (and R finite) then *every* member of the point spectrum of (3.5) is real *and non-zero*. Potential bifurcation points are therefore obtained from the eigenvalue problem (3.3). Let

$$\begin{pmatrix} v(x,y,z) \\ p(x,y,z) \end{pmatrix}=e^{\lambda x}\left[\begin{pmatrix} v_1(y) \\ p_1(y) \end{pmatrix}\cos\beta z+\begin{pmatrix} v_2(y) \\ p_2(y) \end{pmatrix}\sin\beta z\right], \tag{3.6}$$

then $(\frac{d^2}{dy^2}+(\lambda^2-\beta^2))p_j+2\lambda U_y v_j=0$ $(j=1,2)$ and

$$\left[\frac{d^2}{dy^2}+(\lambda^2-\beta^2)\right]^2 v_j-\lambda R(U-c)\left[\frac{d^2}{dy^2}+(\lambda^2-\beta^2)\right]v_j+\lambda RU_{yy}v_j=0 \tag{3.7}$$

which is a modified (real) form of the three-dimensional Orr-Sommerfeld equation. Given (c, R, β) (all real) there are *two* linearly independent eigenfunctions for each eigenvalue λ of (3.7), hence each eigenvalue of (3.7) has (generically) geometric and algebraic multiplicity two. Consequently if there exists an eigenvalue λ of (3.7) with $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\lambda) \neq 0$ then it is (generically) double and with its complex conjugate, the associated 4 eigenfunctions form a basis for a four-dimensional (spatial) centre-subspace.

Hopf bifurcation points are found by fixing (c, β) and increasing R until there exists an eigenvalue of (3.7) with $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\lambda) \neq 0$, or for fixed β the neutral curve in the (c, R) plane can be obtained. When $\beta = 0$ the neutral curve is as shown in Figure 2.1 and when $\beta \neq 0$ Squire's theorem (Drazin & Reid [1981, p. 155]) can be adapted to determine the $\beta \neq 0$ neutral curve: suppose $\beta = 0$ and $\lambda = i\alpha$ $(\alpha \in \mathbb{R})$ and let (c, R_o) be a point on the neutral curve with wavenumber α_o. Then for $\beta \neq 0$ Squire's theorem states that for each given c the neutral point (for $\beta \neq 0$) is shifted (positively) to $R_\beta = R_o\alpha_o/\alpha_\beta$ where $\alpha_\beta = \sqrt{\alpha_o^2 - \beta^2}$ (assuming $|\beta| < |\alpha_o|$). Given the neutral curve for $\beta = 0$ it is therefore straightforward to construct the $\beta \neq 0$ neutral surface and curves for various values of β are shown in Figure 3.1.

Note that for each admissable β there exists a point in (c, R) space at which pure imaginary eigenvalues for the 2D and 3D states exist simultaneously (where the $\beta = 0$ and $\beta \neq 0$ curves intersect). These points are codimension 2 bifurcation points and since the eigenvalues are generically nonresonant, they correspond to points of bifurcation to spatially quasi-periodic states and are analyzed in Section 4. For *particular* values of β the eigenvalues of the codimension 2 points will be resonant (this will be of codimension 3). For example if at $\beta = \beta_o$ the 2D and 3D eigenvalues lie at α_o and $\frac{1}{2}\alpha_o$ the resonance is a spatial version of the Craik resonant triad (Craik [1971,1985]). In Section 4 these resonances (α_o and α_o/n $n = 2, 3, 4$) are considered from the spatial point of view; that is, they are codimension *three* organizing centers for more complex spatial structure. In this section it is assumed that (c, β) take generic (admissable) values and the codimension 1 bifurcations associated with variation of R are treated.

We now proceed to compute the bifurcation from the neutral curve with $\beta \neq 0$. Choose admissable values of β and c and suppose $R = R_o$ is the neutral point with eigenvalue $\lambda = i\alpha_o$ with $\alpha_o \in \mathbf{R}$. Scale $x \mapsto \alpha x$ so that α appears as a coefficient in the nonlinear set of equations (3.1). Reverting to complex coordinates the solution of the linear equation (3.2) at the neutral point is given by

$$\left.\begin{aligned} &\begin{pmatrix} u_1(x,y,z) \\ v_1(x,y,z) \\ p_1(x,y,z) \end{pmatrix} = 2\mathrm{Re}\left[(Ae^{i(x+\beta z)} + Be^{i(x-\beta z)}) \begin{pmatrix} \hat{u}_1(y) \\ \hat{v}_1(y) \\ \hat{p}_1(y) \end{pmatrix} \right] \\ &\text{and} \\ &\qquad w_1(x,y,z) = 2\mathrm{Re}\left[(Ae^{i(x+\beta z)} - Be^{i(x-\beta z)}) \hat{w}_1(y) \right] \end{aligned}\right\} \tag{3.8}$$

where $A, B \in \mathbf{C}$ are complex amplitudes. Let $\hat{\Delta} \stackrel{\text{def}}{=} \frac{d^2}{dy^2} - (\alpha_o^2 + \beta^2)$ then the complex functions $(\hat{u}_1, \hat{v}_2, \hat{w}_3, \hat{p}_4)$ satisfy

$$\mathbf{L}_4(\alpha_o, \beta) \cdot \hat{v}_1 \stackrel{\text{def}}{=} \hat{\Delta}^2 \hat{v}_1 + i\alpha_o R U_{yy} \hat{v}_1 - i\alpha_o R(U - c)\hat{\Delta}\hat{v}_1 = 0 \tag{3.9}$$

$$\left.\begin{aligned} \hat{u}_1 &= \frac{i\alpha_o}{\alpha_o^2 + \beta^2} \frac{\partial \hat{v}_1}{\partial y} + \frac{\beta^2 R}{\alpha_o^2 + \beta^2} \mathbf{L}_2^{-1}(U_y \hat{v}_1) \\ \hat{w}_1 &= \frac{i\beta}{\alpha_o^2 + \beta^2} \frac{\partial \hat{v}_1}{\partial y} - \frac{\alpha_o \beta R}{\alpha_o^2 + \beta^2} \mathbf{L}_2^{-1}(U_y \hat{v}_1) \\ \hat{p}_1 &= \frac{1}{\alpha_o^2 + \beta^2} (i\alpha_o U_y \hat{v}_1 + \frac{1}{R} \mathbf{L}_2 (\frac{\partial \hat{v}_1}{\partial y})) \end{aligned}\right\} \tag{3.10}$$

where $\mathbf{L}_2 \stackrel{\text{def}}{=} \hat{\Delta} - i\alpha_o R(U - c)$.

The idea is to apply the centre-manifold theorem to the spatial bifurcation of periodic states. Rigorous application of the centre-manifold theory is not possible in this case due to the fact that the Blasius solution does not satisfy the Navier-Stokes equations and the further neglect of non-parallel terms. We can however give a formal construction of the centre-manifold using the theory of Coullet & Spiegel [1983]. There are two steps in the reduction to normal form. Suppose (3.1) has been recast as an evolution equation (this is carried out for the primitive variables in Section 3.5),

$$\frac{\partial}{\partial x} \Phi = \mathbf{L}(c, R) \cdot \Phi + \mathbf{N}(\Phi, u, ; R). \tag{3.11}$$

For admissable c and $R = R_o$ the linear operator $\mathbf{L}$ satisfies

$$\mathbf{L}(c, R) \cdot \hat{\Phi}(y, z) = i\alpha_o \hat{\Phi}(y, z) \qquad \alpha_o \in \mathbf{R}$$

where $\hat{\Phi}(y, z) = (Ae^{i\beta z} + Be^{-i\beta z})\hat{\Phi}_1(y)$ with $A, B \in \mathbf{C}$. Suppose that $\lambda = \pm i\alpha_o$ are the only eigenvalues of $\mathbf{L}(c, R_o)$ on the imaginary axis. Then Φ can be expressed as

$$\begin{aligned}\Phi(x, y, z) &= (A(x)e^{i\beta z} + B(x)e^{-i\beta z})\hat{\Phi}_1(y) \\ &\quad + (\overline{A(x)}e^{-i\beta z} + \overline{B(x)}e^{i\beta z})\overline{\hat{\Phi}_1(y)} + \Psi(x, y, z) \end{aligned} \tag{3.12}$$

where Ψ consists of all modes not in the center subspace. The first step in the centre-manifold reduction is to substitute (3.12) into (3.11) (at $R = R_o$) resulting in

$$\left.\begin{aligned} \frac{dA}{dx} &= f_1(A, B, \overline{A}, \overline{B}, \Psi) \\ \frac{dB}{dx} &= f_2(A, B, \overline{A}, \overline{B}, \Psi) \end{aligned}\right\} \tag{3.13}$$

$$\frac{d\Psi}{dx} = f_3(A, B, \overline{A}, \overline{B}, \Psi) \tag{3.14}$$

with additional equations for $\overline{A}$ and $\overline{B}$. All eigenvalues of $df_3(0, 0, 0, 0, 0)$ are off the imaginary axis and presumably (3.14) can be solved for Ψ as a function of $A, B, \overline{A}, \overline{B}$. Back substitution of Ψ into (3.13) results in a vectorfield on $\mathbf{C}^2$. The reduced vectorfield will be an O(2)-equivariant vectorfield on $\mathbf{C}^2$ with $\pm i\alpha_o$ (double) eigenvalues at the linearization. Normal form theory is then applied to transform (3.13) to an $\mathbf{O}(2) \times \mathbf{S}^1$-equivariant vectorfield on $\mathbf{C}^2$;

$$\begin{aligned} \frac{dA}{dx} &= Af_1(\alpha - \alpha_o, R - R_o, |A|^2, |B|^2) \\ \frac{dB}{dx} &= Bf_2(\alpha - \alpha_o, R - R_o, |A|^2, |B|^2) \end{aligned} \tag{3.15}$$

but the $\mathbf{O}(2) \times \mathbf{S}^1$ symmetry requires $f_2(\cdot, \cdot, |A|^2, |B|^2) = f_1(\cdot, \cdot, |B|^2, |A|^2)$.

First we will analyze the normal form equations (3.15) using the theory of Golubitsky, Stewart & Schaeffer [1988, Chapter XVII] and Golubitsky & Roberts [1987] keeping in mind that the "frequency" is in fact the wavenumber α. Then details of the construction of (3.15) are given.

Let $N = |A|^2 + |B|^2$, $\Delta = \delta^2$ and $\delta = |B|^2 - |A|^2$ then following Golubitsky & Roberts [1985, prop. 2.1] the set (3.15) can be transformed to

$$\begin{aligned} \dot{A} &= (p + iq + (r + is)\delta)A \\ \dot{B} &= (p + iq - (r + is)\delta)B \end{aligned} \tag{3.16}$$

where p, q, r and s are $\mathbf{O}(2) \times \mathbf{S}^1$ invariant functions; that is, they are functions of N, Δ and parameters. Writing $A = ae^{i\psi_1}$ and $B = be^{i\psi_2}$ the set (3.16) can be split into amplitude/phase equations

$$\left.\begin{aligned} \dot{a} &= (p + r\delta)a \\ \dot{b} &= (p - r\delta)b \end{aligned}\right\} \quad \text{amplitude equations} \tag{3.17}$$

$$\left.\begin{aligned} \dot{\psi}_1 &= (q + s\delta) \\ \dot{\psi}_2 &= (q - s\delta) \end{aligned}\right\} \quad \text{phase equations.} \tag{3.18}$$

The idea is to solve the phase equations for $\alpha - \alpha_o$. Then substitution of the expression for $\alpha - \alpha_o$ into (3.17) results in a function of $R - R_o$ and (a, b) alone (when (c, β) are fixed):

$$\frac{d}{dx}\begin{pmatrix} a \\ b \end{pmatrix} = g(a, b, R - R_o)$$

where

$$g(a, b, R - R_o) = p(N, \Delta, R - R_o)\begin{pmatrix} a \\ b \end{pmatrix} + r(N, \Delta, R - R_o)\delta\begin{pmatrix} a \\ -b \end{pmatrix}. \tag{3.19}$$

Generically there are two types of solutions of the normal form: (a) oblique waves with $a \neq 0$ and $b = 0$ which correspond to waves with a wavefront at some angle to the streamwise direction and (b) "standing waves" with $a = b$. These correspond to waves that travel in the streamwise direction but are periodic in the spanwise direction. There is also the possibility (with an additional parameter) for the two classes of waves to interact producing quasi-periodic waves. Further details on the symmetries and more complete analysis of the normal form (3.16)-(3.19) can be found in Chapter XVII of Golubitsky, Stewart & Schaeffer [1988].

The coefficients in the normal form (3.12) are obtained in the following way. The eigenfunction $\hat{\Phi}_1(y)$ in (3.12) is easily constructed using the expressions in (3.8)-(3.10). The equation (3.14) is solved only to sufficient order (in A and B) to obtain the generic normal form. This is done by expanding Ψ as a quadratic polynomial in $A, B, \overline{A}$ and $\overline{B}$. Dropping terms that don't appear in the normal form, Ψ_2 is constructed from (u_2, v_2, w_2) where

$$
\begin{aligned}
v_2(x,y,z) = 2\mathrm{Re}\Big[&(A^2 e^{2i(x+\beta z)} + B^2 e^{2i(x-\beta z)})\hat{v}_{21}(y) + 2ABe^{2ix}\hat{v}_{22}(y)\Big] \\
&+ (AB^* e^{2i\beta z} + A^* B e^{-2i\beta z})\hat{v}_{23}(y)
\end{aligned}
\tag{3.20}
$$

where

$$
\left.
\begin{aligned}
\frac{1}{R_o}\mathrm{L}_4(2\alpha_o, 2\beta)\cdot\hat{v}_{21} &= -2\frac{d}{dy}\left(\frac{d\hat{v}_1}{dy}\frac{d\hat{v}_1}{dy} - \hat{v}_1\frac{d^2 v_1}{dy^2}\right) \\
\frac{1}{R_o}\mathrm{L}_4(2\alpha_o, 0)\cdot\hat{v}_{22} &= 4\alpha_o^2\frac{d}{dy}(\hat{u}_1^2 - \hat{v}_1^2) - 2i\alpha_o\left(\frac{d}{dy^2} + 4\alpha_o^2\right)\hat{u}_1\hat{v}_1 \\
\frac{1}{R_o}\mathrm{L}_4(0, 2\beta)\cdot\hat{v}_{23} &= -8\beta^2\frac{d}{dy}(|\hat{v}_1|^2 + |\hat{w}_1|^2) - 2i\beta\left(\frac{d^2}{dy^2} + 4\beta^2\right)(\hat{v}_1^*\hat{w}_1 - \hat{v}_1\hat{w}_1^*).
\end{aligned}
\right\}
\tag{3.21}
$$

For the streamwise and spanwise velocity we find

$$
\begin{aligned}
u_2(x,y,z) = 2\mathrm{Re}\Big[&(A^2 e^{2i(x+\beta z)} + B^2 e^{2i(x-\beta z)})\hat{u}_{21}(y) + 2ABe^{2ix}\hat{u}_{22}(y)\Big] \\
&+ (AB^* e^{2i\beta z} + A^* B e^{-2i\beta z})\hat{u}_{23}(y) + (|A|^2 + |B|^2)\hat{u}_{24}(y) \qquad (3.22) \\
w_2(x,y,z) = 2\mathrm{Re}\Big[&(A^2 e^{2i(x+\beta z)} - B^2 e^{2i(x-\beta z)})\hat{w}_{21}(y)\Big] \\
&+ (AB^* e^{2i\beta z} - A^* B e^{-2i\beta z})\hat{w}_{23}(y) + (|A|^2 - |B|^2)\hat{w}_{24}(y) \qquad (3.23)
\end{aligned}
$$

where

$$\left.\begin{aligned}
-\frac{d\hat{v}_{21}}{dy} &= 2i\alpha_o\hat{u}_{21}(y) + 2i\beta\hat{w}_{21}(y), \\
\hat{u}_{22} &= \frac{i}{2\alpha_o}\frac{d}{dy}\hat{v}_22\,, \qquad \hat{w}_{23}(y) = \frac{i}{2\beta}\frac{d}{dy}\hat{v}_{23}, \\
\left(\frac{d^2}{dy^2} - 4\beta^2\right)\hat{u}_{23}(y) &= RU_y\hat{v}_{23} + R\frac{d}{dy}(\hat{u}_1\hat{v}_1^* + \hat{u}_1^*\hat{v}_1) - 2i\beta R(\hat{u}_1\hat{w}_1^* - \hat{u}_1^*\hat{w}_1), \\
\frac{d}{dy}\hat{u}_{24}(y) &= R(\hat{u}_1\hat{v}_1^* + \hat{u}_1^*\hat{v}_1) \\
\text{and} \\
\frac{d}{dy}\hat{w}_{24}(y) &= R(\hat{w}_1\hat{v}_1^* + \hat{w}_1^*\hat{v}_1).
\end{aligned}\right\} \tag{3.24}$$

Substitution of (3.20)-3.24) into the vectorfield (3.13) and subsequent normalization results in

$$\begin{aligned}
\frac{dA}{dx} &= A\left[h_{31}(R-R_o) + h_{32}(\alpha-\alpha_o) + h_{33}|A|^2 + h_{34}|B|^2 + \cdots\right] \\
\frac{dB}{dx} &= A\left[h_{31}(R-R_o) + h_{32}(\alpha-\alpha_o) + h_{34}|A|^2 + h_{33}|B|^2 + \cdots\right]
\end{aligned} \tag{3.25}$$

where $h_{3j} = [\overline{\phi_*}, \hat{h}_{3j}(y)]$ and ϕ_* is the adjoint eigenfunction of the Orr-Sommerfeld equation and

$$\left.\begin{aligned}
\hat{h}_{31}(y) &= \frac{1}{R_o}\hat{\Delta}\hat{\Delta}\hat{v}_1, \\
\hat{h}_{32}(y) &= -\frac{2i\alpha_o}{R_o}\hat{\Delta}\hat{v}_1 - \frac{2}{R_o}\hat{\Delta}\frac{d\hat{u}_1}{dy} + 2U_y\left(\frac{d\hat{v}_1}{dy} + i\alpha_o\hat{u}_1\right) \\
&\quad + U_{yy}\hat{v}_1 + (U-c)\hat{\Delta}\hat{v}_1 + 2i\alpha_o(U-c)\frac{d\hat{u}_1}{dy} \\
\hat{h}_{33}(y) &= \left[\frac{d^2}{dy^2} + (\alpha_o^2 + \beta^2)\right]k_{11}(y) + \frac{d}{dy}k_{12}(y) \\
\hat{h}_{34}(y) &= \left[\frac{d^2}{dy^2} + (\alpha_o^2 + \beta^2)\right]k_{21}(y) + \frac{d}{dy}k_{22}(y).
\end{aligned}\right\} \tag{3.26}$$

The operator $\hat{\Delta}$ is the reduced Laplacian, $\hat{\Delta} \stackrel{\text{def}}{=} \frac{d^2}{dy^2} - (\alpha_o^2 + \beta^2)$ and the functions k_{ij} are defined in

$$k_{11} = (-i\alpha_o\hat{u}_1^* - i\beta\hat{w}_1^*)\hat{v}_{21} - (i\alpha_o\hat{u}_{24} + i\beta\hat{w}_{24})\hat{v}_1$$

$$- \tfrac{1}{2}\hat{v}_1^*(2i\alpha_o\hat{u}_{21} + 2i\beta\hat{w}_{21}) \tag{3.27a}$$

$$\begin{aligned} k_{21} &:= (-i\alpha_o\hat{u}_1 + i\beta\hat{w}_1)\hat{v}_{23} + 2\hat{v}_{22}(i\beta\hat{w}_1^* - i\alpha_o\hat{u}_1^*) - 2i\alpha_o\hat{v}_1^*\hat{u}_{22} \\ &\quad + \hat{v}_1(i\beta\hat{w}_{24} - i\alpha_o\hat{u}_{24}) - \hat{v}_1(i\alpha_o\hat{u}_{23} + i\beta\hat{w}_{23}) \end{aligned} \tag{3.27b}$$

$$\begin{aligned} k_{12} &:= -2(i\alpha_o\hat{u}_1 + i\beta\hat{w}_1)(i\alpha_o\hat{u}_{21} + i\beta\hat{w}_{24}) - 2(\alpha_o^2 + \beta^2)\hat{v}_1^*\hat{v}_{21} \\ &\quad + 2(-i\alpha_o\hat{u}_1^* - i\beta\hat{w}_1^*)(i\alpha_o\hat{u}_{21} + i\beta\hat{w}_{21}) \end{aligned} \tag{3.27c}$$

$$\begin{aligned} k_{22} &:= -2(i\alpha_o\hat{u}_1 + i\beta\hat{w}_1)(i\alpha_o\hat{u}_{21} - i\beta\hat{w}_{21}) - 2(i\alpha_o\hat{u}_1 - i\beta\hat{w}_1)(i\alpha_o\hat{u}_{23} + i\beta\hat{w}_{23}) \\ &\quad - 4i\alpha_o\hat{u}_{22}(i\alpha_o\hat{u}_1^* - i\beta\hat{w}_1^*) - 2(\alpha_o^2 + \beta^2)(\hat{v}_1\hat{v}_{23} + 2\hat{v}_1^*\hat{v}_{22}) \end{aligned} \tag{3.27d}$$

The vectorfield in (3.25) can be recast as

$$\frac{dA}{dx} = A\left[h_{31}(R - R_o) + h_{32}(\alpha - \alpha_o) + \tfrac{1}{2}(h_{33} + h_{34})N + \tfrac{1}{2}(h_{34} - h_{33})\delta + \cdots\right]$$

which is of the form (3.16) with

$$\begin{aligned} p + iq &= h_{31}(R - R_o) + h_{32}(\alpha - \alpha_o) + \tfrac{1}{2}(h_{33} + h_{34})H + \cdots \\ r + is &= \tfrac{1}{2}(h_{31} - h_{33}) + \cdots . \end{aligned} \tag{3.28}$$

It is now straightforward to apply the theory for $\mathbf{O(2)} \times \mathbf{S}^1$ equivariant normal forms to (3.28) given the complex numbers h_{3j} $j = 1, \ldots, 4$. Numerical evaluation of the coefficients in (3.28) is considered is Section 3.2.

It is important to note that the above bifurcations are spatial bifurcations and the stability assignments given in Golubitsky, Stewart & Schaeffer [1988, Chapt. XVII] are not applicable. To determine the stability of the two classes of waves (oblique and standing) time will have to be reintroduced *and* the possibility of sideband instability considered. This is a very interesting problem that we will treat in detail elsewhere.

3.2 Computation of the coefficients for spatial Hopf bifurcation

In this Section details of the numerical evaluation of the coefficients in the $\mathbf{O(2)}\times\mathbf{S}^1$ equivariant normal form obtained in Section 3.1 are given for the case where the equilibrium state is the Blasius boundary layer. Numerical evaluation is essential because the

Blasius solution is given implicitly as the solution of a differential equation. The basic problem is to construct the functions $\hat{h}_{3j}(y)$ $(j = 1, \ldots, 4)$ in equation (3.26) which in turn depend on complex functions that satisfy the homogeneous or inhomogeneous Orr-Sommerfeld (or related) equation. The Blasius equation, the Orr-Sommerfeld equation and related equations are all posed on the semi-infinite interval $y \in [0, \infty)$. However numerical solution of the Orr-Sommerfeld equation on semi-infinite domains is treated in Bridges & Morris [1984b,1987] and we use their basic algorithm here. The semi-infinite domain is mapped to $[-1, +1]$ using the algebraic transformation

$$Y = \frac{y-2}{y+2} \qquad y \in [0, \infty). \tag{3.29}$$

Then $\frac{d}{dy} \mapsto m(Y)\frac{d}{dY}$ where $m(Y) = (1-Y)^2/4$ and all functions of y are considered as functions of Y and expanded in finite series of Chebyshev polynomials. As an example we consider the construction of a finite Chebyshev series expansion for the complex function $\hat{v}_{21}(y)$ (in equation (3.19)) which satisfies an inhomogeneous (modified) Orr-Sommerfeld equation. Mapping $y \mapsto Y$ the governing equation for $\hat{v}_{21}$ is

$$\begin{aligned} m(Y)\frac{d\hat{v}_{21}}{dY} &= \hat{\xi} \\ m\frac{d}{dY}(m\frac{d}{dY}(m\frac{d\hat{\xi}}{dY})) + \hat{A}(Y)m\frac{d\hat{\xi}}{dY} + \hat{B}(Y)\hat{v}_{21} &= \\ 2R_o m\frac{d}{dY}\left[m\frac{d}{dY}(\hat{v}_1\frac{d}{dY}\hat{v}_1) - 2(m\frac{d\hat{v}_1}{dY})^2\right] \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} \hat{A}(Y) &= -2(\alpha_o^2 + \beta^2) - i\alpha_o R_o(U(Y) - c) \\ \hat{B}(Y) &= (\alpha_o^2 + \beta^2)^2 + i\alpha_o R_o(\alpha_o^2 + \beta^2)(U - c) + i\alpha_o R_o m\frac{d}{dY}(m\frac{dU}{dY}) \end{aligned} \tag{3.31}$$

The first order vertical velocity is obtained as a solution of the Orr-Sommerfeld equation as in Bridges & Morris [1987]. The metric $m(Y)$ is expanded in a series of Chebyshev polynomials,

$$m(Y) = \tfrac{1}{2}m_0 + m_1\mathrm{T}_1(Y) + m_2\mathrm{T}_2(Y) \tag{3.32}$$

with $m_0 = \frac{3}{4}$, $m_1 = -\frac{1}{2}$ and $m_2 = \frac{1}{8}$. Then the Chebyshev product and differentiation formulae can be used to write the right hand side of (3.30) in a finite series of Chebyshev polynomials. Similarily the coefficients $\hat{A}(Y)$ and $\hat{B}(Y)$ can be expanded in finite seres of Chebyshev polynomials. It is then straightforward to expand $\hat{v}_{21}(Y)$ in a finite series. Substitution into (3.30) and application of the Chebyshev$-\tau$ method results in a discrete system. After numerically eliminating $\hat{\xi}$ (3.30) reduces to the finite dimensional matrix equation

$$[\mathbf{D}_4(2\alpha_o, 2\beta)] \cdot \{\hat{v}_{21}\} = \{\text{rhs}\}. \tag{3.33}$$

Assuming $(2\alpha_o, 2\beta)$ is not an eigenvalue when (α_o, β) is (non-resonance point) the (complex) system (3.33) is easily inverted to find the vector $\{\hat{v}_{21}\}$ of Chebyshev coefficients. The remainder of the second order functions are obtained in a similar fashion. Then the functions $\hat{h}_{3j}(y)$ are obtained by repeated use of the Chebyshev product and differentiation formulae. The matrix $[\mathbf{D}_4(\alpha_o, \beta)]$ is the discretized Orr-Sommerfeld equation and therefore (when $i\alpha_o$ is and eigenvalue) has a left eigenvector. Instead of using the adjoint eigenfunction of the continuous Orr-Sommerfeld equation to obtain the bifurcation coefficients we simply use the left eigenvector of the matrix $[\mathbf{D}_4(\alpha_o, \beta)]$. Let $\{\phi_*\}$ be the left eigenvector of $\mathbf{D}_4$; that is,

$$\{\phi_*\}^H \cdot [\mathbf{D}_4(\alpha_o, \beta)] = 0, \tag{3.34}$$

then the bifurcation coefficients h_{3j} are easily obtained by taking the (discrete) complex inner-product of $\{\phi_*\}$ with $\{\hat{h}_{3j}\}$. Complete details of the numerical calculations will be reported in Bridges [1991b].

Alternatively a complete analysis of transition process (through the spatial evolution equation) can be carried out in the discrete setting. Write the Navier-Stokes equations as an evolution equation in x: $\frac{\partial}{\partial x}\Phi = \mathbf{L}(c, R) \cdot \Phi + \mathbf{N}(\Phi; R)$ and for brevity we'll sketch the 2D case. Eliminate the differential operators in y by expanding Φ is a finite series of Chebyshev polynomials:

$$\Phi(x, y) = \tfrac{1}{2}\Phi_0(x) + \sum_{n=1}^{N} \Phi_n(x) T_n(y). \tag{3.35}$$

The evolution equation for Φ is then reduced to an evolution equation on the finite (although large) dimensional space $\mathbf{R}^{IN}$,

$$\frac{\partial}{\partial x}\Psi = \hat{\mathbf{L}}(c,R)\cdot\Psi + \hat{\mathbf{N}}(\Psi,R) \tag{3.36}$$

where $\Psi = (\Phi_0, \Phi_1, \ldots)^T$, $\hat{\mathbf{L}}$ is a matrix and $\hat{\mathbf{N}}$ is an algebraic nonlinear operator. It is then straightforward to apply the usual bifurcation theory for evolution equations in finite dimensions to (3.36).

3.3 Secondary bifurcations 2D → 3D

An alternate route for three-dimensionality to arise is through a secondary bifurcation from a 2D state to a 3D state. In fact this is a widely accepted theory for the origin of three-dimensionality (Orszag & Patera [1983], Bayly, Orszag & Herbert [1988], Herbert [1988]). The "secondary instability" theories of Orszag & Patera and Herbert are however temporal theories. In this section we introduce a spatial theory for the bifurcation from 2D finite amplitude states to three-dimensional states. The idea is to study the *spatial* Floquet multipliers along a branch of *spatially* periodic 2D states but with 3D perturbations.

Let (u, v, p) be a 2D spatially periodic state satisfying (2.1) and consider the addition of a 3D perturbation $(u+\xi, v+\eta, w, p+q)$. Substitution into the 3D Navier-Stokes equations (3.1) and linearization about the 2D state results in the following system with periodic (in x) coefficients

$$\frac{\partial\xi}{dx} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{dz} = 0 \tag{3.37}$$

$$\left[(u+U-c)\frac{\partial}{\partial x} - \frac{1}{R}\Delta\right]\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} + \nabla q + \begin{bmatrix} u_x + v\frac{\partial}{\partial y} & U_y + u_y & 0\\ v_x & v_y + v\frac{\partial}{\partial y} & 0\\ 0 & 0 & v\frac{\partial}{\partial y}\end{bmatrix}\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} = 0. \tag{3.38}$$

The pressure perturbation q can be eliminated by taking the divergence of the perturbation momentum equations,

$$\Delta q = -2\left[(U_y + u_y)\eta_x + v_y\eta_y + u_x\xi_x + v_x\xi_y\right]. \tag{3.39}$$

Now, taking the Laplacian of the streamwise and vertical velocity perturbation equations results in the following coupled linear equations for (ξ, η) with periodic (in x) coefficients,

$$\begin{aligned}
&\Delta\Delta\xi - R\frac{\partial}{\partial x}\Delta q - R\Delta\left\{(U_y + u_y)\eta + (u + U - c)\xi_x + \xi u_x + v\xi_y\right\} = 0 \\
&\Delta\Delta\eta - R\frac{\partial}{\partial y}\Delta q - R\Delta\left\{v_y\eta + (u + U - c)\eta_x + \xi v_x + v\eta_y\right\} = 0
\end{aligned} \tag{3.40}$$

with Δq given by (3.39) as a function of (ξ, η). There is also an additional decoupled eigenvalue problem associated with the spanwise velocity perturbation; in particular, given (ξ, η), q is obtained from (3.39) and the spanwise velocity perturbation is given by the solution of

$$-\frac{1}{R}\Delta w + (u + U - c)\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = -\frac{\partial q}{\partial z}. \tag{3.41}$$

Equations (3.40) and (3.41) *together* form the eigenvalue problem for the secondary bifurcations. But, since they are decoupled the Floquet eigenvalue problem associated with (3.41) can be studied independently. Let

$$w(x, y, z) = e^{\gamma x}\left[w_1(x, y)\cos\beta z + w_2(x, y)\sin\beta z\right],$$

then $w_j(x, y)$ $(j = 1, 2)$ satisfy the following *quadratic in the parameter* eigenvalue problem,

$$\begin{aligned}
\gamma^2 w_j + \gamma&\left\{2\frac{\partial w_j}{\partial x} - R(u + U - c)w_j\right\} \\
&+ \left\{\frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y^2} - \beta^2 w_j - R(u + U - c)\frac{\partial w_j}{\partial x} - Rv\frac{\partial w_j}{\partial y}\right\} = 0
\end{aligned} \tag{3.42}$$

$(j = 1, 2)$ for the spatial Floquet exponent γ. Given expressions for the periodic functions $u(x, y)$ and $v(x, y)$ along a branch of 2D spatially periodic states, the eigenvalue problem (3.42) can be discretized and solved using the methods described in Section 2.2.

The main eigenvalue problem for the spatial secondary "instability" is the coupled set (3.40). Since the coefficients of (3.40) are periodic in x and independent of z we can take

$$\begin{pmatrix}\xi(x, y, z)\\ \eta(x, y, z)\end{pmatrix} = e^{\gamma x}\left[\begin{pmatrix}\xi_1(x, y)\\ \eta_1(x, y)\end{pmatrix}\cos\beta z + \begin{pmatrix}\xi_2(x, y)\\ \eta_2(x, y)\end{pmatrix}\sin\beta z\right] \tag{3.43}$$

with (ξ_j, η_j) $(j = 1, 2)$ periodic in x. Substitution of (3.43) into (3.40) results in the nonlinear (of degree 4) in the parameter eigenvalue problem

$$\mathbf{L}(\epsilon,\beta)\cdot\begin{pmatrix}\xi_j\\ \eta_j\end{pmatrix} = \sum_{j=0}^{4}\gamma^{4-j}\mathbf{L}_j(\epsilon,\beta)\cdot\begin{pmatrix}\xi_j\\ \eta_j\end{pmatrix} = 0. \qquad (3.44)$$

Explicit expressions for the operators $\mathbf{L}_j(\epsilon, \beta)$ are easily obtained by substituting (3.43) into (3.39) and (3.40). The parameter ϵ represents a parametrization of the branch of 2D states and β is the spanwise wavenumber of the perturbation. In particular, the Floquet exponents $\gamma(\epsilon, \beta)$ will depend on *two parameters* which will result in potentially more complex bifurcations (when compared with the 2D secondary bifurcation problem in Section 2.2 where $\beta = 0$). The complexity of the bifurcations can also be increased by changing parameters that alter the equilibrium state such as c (the wavespeed) or using Falkner-Skan flows rather than the Blasius flow for the equilibrium state or adding a compliant wall along the boundary (Carpenter & Garrad [1985] & Carpenter & Morris[1990]).

The types of secondary bifurcation points to be expected from the eigenvalue problem (3.44) will be similar to those described in Section 2.2 (wavelength doubling and secondary bifurcation to (streamwise) quasi-periodic states) but with the addition of spanwise periodicity of wavenumber β (which is an *independent* parameter).

To determine values of the parameters (ϵ, β) at which secondary bifurcations occur would require numerical solution of the nonlinear eigenvalue problem (3.44). One way to show that in fact secondary bifurcations to spatially quasi-periodic states are to be expected is to introduce a second parameter whose variation brings the secondary bifurcation point down to the origin forming a codimension 2 singularity. Then secondary bifurcation to spatially quasi-periodic states can be found in the unfolding. In Section 4 it is shown that for each β (sufficiently small) there exists a codimension 2 point in (c, R) space whose unfolding contains secondary bifurcations to spatially quasi-periodic states. In fact all along the *upper* branch of the 2D neutral curve secondary bifurcation to quasi-periodic states (with periodic spanwise variation) will be prevalent.

Symmetry will play an important role in the secondary bifurcations. For the secondary bifurcation to streamwise quasi-periodic states the normal form will be $O(2) \times S^1$-equivariant with the $O(2)$ action associated with the spanwise periodicity and reflection and the S^1 action is associated with the second streamwise wavenumber (assuming the Floquet multiplier lies at an irrational point on the unit circle). A torus bifurcation in the continuous system is locally equivalent to a Hopf bifurcation in a map (discrete time system). In essence *the above secondary bifurcation is equivalent to a Hopf bifurcation in an* $O(2)$*-equivariant map.* Results on Hopf bifurcation in maps have been obtained by Chossat & Golubitsky [1988, p. 1262]. Modulo some (substantial) technical details the $O(2)$-equivariant Hopf bifurcation in maps resembles the $O(2)$-equivariant Hopf bifurcation in continuous systems. In particular *there will be two classes of secondary quasi-periodic states.* This is easy to see physically: the secondary state can correspond to an oblique wave (where the streamwise wavenumber does not resonate with the 2D wavenumber) or the secondary state can be oriented in the streamwise direction (but spanwise periodic and again non-resonant). The group-theoretic results of Chossat & Golubitsky can be used to obtain further information. There will be a group orbit of quasi-periodic states; in particular two oblique waves with isotropy subgroup $\widetilde{SO}(2)$ and a continuous group orbit of "standing" secondary states (a torus of invariant tori!) with discrete isotropy subgroup. With the addition of another parameter there will also exist points of tertiary bifurcation to 3-tori! The analysis of secondary bifurcation to quasi-periodic states with symmetry is an interesting area for further study. Our analysis in Section 4 shows that this bifurcation will play a crucial role along the upper branch of the 2D neutral curve in shear flows.

The other class of secondary bifurcations is wavelength doubling (spatial Floquet multiplier passing through -1). Suppose the streamwise wavenumber of the basic state is normalized to 1 (wavelength 2π) and that a Floquet multiplier lies at -1. To study the bifurcation of 4π-periodic states we use the equations (3.1) perturbed about the 2D state. On the space of 4π-periodic functions however the nonlinear problem is Z_2^ρ-equivariant with action $\rho \cdot f(x, y, z) = f(x + 2\pi, y, z)$. The non-trivial spanwise variation however provides an additional $O(2)$-equivariance of the nonlinear equations.

Therefore *secondary bifurcation to wavelength doubling from 2D states to 3D states with spanwise periodicity is a $\mathbf{Z}_2^R \times \mathrm{O}(2)$ equivariant bifurcation problem*, or in terms of dynamical systems theory, the above bifurcation corresponds to *period-doubling in the presence of a continuous symmetry*. Period doubling with a continuous symmetry is a difficult problem. Acting on the periodic orbit with the $\mathbf{SO}(2) \subset \mathbf{O}(2)$ action results in a torus of periodic orbits. Period-doubling will therefore correspond to a doubling of the whole manifold of periodic orbits (see Vanderbauwhede [1989,1990]). In light of its importance for spatial bifurcations in shear flows the properties of period-doubling with $\mathbf{O}(2)$ symmetry is an interesting area for further research.

3.4 Secondary bifurcations 3D → 3D

Secondary bifurcations from 3D states that are periodic in the streamwise and spanwise direction are of great importance in shear flows. Three dimensionality is essential for true turbulence and the theory in Sections 3.1 and 3.3 presents two routes to 3D states with spanwise periodicity. 3D states with spanwise periodicity are apparently an inevitable stage in the transition process. If a (spanwise periodic) 3D state exists and the streamwise flow is also periodic (it can either be the basic periodic state or have wavelength doubled (or n-tupled) any number of times), then we can apply two-dimensional spatial Floquet theory to study tertiary bifurcations in *both* the streamwise *and* spanwise directions.

Linearization of the set of equations (3.1) about a 3D state periodic in (x, z) results in a pde with doubly-periodic coefficients to which spatial Floquet theory (Eastham [1973, Chapt. 6]) will be applied. A basic question to be addressed in the application of 2D spatial Floquet theory is the role of spanwise spatial bifurcations in the transition process. In particular, is *spanwise periodicity* a good assumption throughout the transition process with bifurcations essentially taking place in the streamwise direction? Or alternatively, does wavelength n-tupling and/or spanwise quasi-periodicity play a significant role in the transition process?

Suppose (u, v, w, p) is a three-dimensional state that is periodic of period $2\pi/\beta$ in the spanwise direction and periodic of period $2\pi n$ ($n \in \mathrm{N}$) in the streamwise direction (it doesn't matter how many times the wavelength has doubled or n-tupled). Introduction of a perturbation about the three-dimensional state $(u+\xi, v+\eta, w+\zeta, p+q)$, substitution into equations (3.1) and linearization about the known 3D state results in the following linear pde with doubly periodic coefficients,

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0 \tag{3.45}$$

$$\begin{aligned} &\left[(u + U - c)\frac{\partial}{\partial x} - \frac{1}{R}\Delta\right]\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} + \nabla q \\ &+ \begin{bmatrix} (u\,v\,w)^T \cdot \nabla + u_x & U_y + u_y & u_z \\ v_x & (u\,v\,w)^T \cdot \nabla + v_y & v_z \\ w_x & w_y & (u\,v\,w)^T \cdot \nabla + w_z \end{bmatrix}\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} = 0. \end{aligned} \tag{3.46}$$

An equation for the pressure perturbation is obtained by taking the divergence of the momentum equations resulting in

$$\Delta q = -2U_y\eta_x - 2\left[\nabla u \cdot \frac{\partial}{\partial x} + \nabla v \cdot \frac{\partial}{\partial y} + \nabla w \cdot \frac{\partial}{\partial z}\right]\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}. \tag{3.47}$$

Taking the Laplacian of each of the momentum equations in (3.46) will eliminate the pressure perturbation via substitution of (3.47) resulting in three linear coupled equations with doubly-periodic coefficient for the velocity perturbations (ξ, η, ζ),

$$\mathrm{L}(x, z) \cdot \begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix} = 0. \tag{3.48}$$

Two-dimensional Floquet theory (Eastham [1973, Chapt. 6]) is easily applied to (3.48). Let

$$\begin{pmatrix}\xi(x, y, z)\\ \eta(x, y, z)\\ \zeta(x, y, z)\end{pmatrix} = e^{\gamma_1 x + \gamma_2 z}\begin{pmatrix}\hat{\xi}(x, y, z)\\ \hat{\eta}(x, y, z)\\ \hat{\zeta}(x, y, z)\end{pmatrix} \tag{3.49}$$

where $\hat{\xi}, \hat{\eta}$ and ζ are doubly periodic of "period" $2\pi n$ in x and $2\pi/\beta$ in z. Substitution of (3.49) into (3.48) results in an eigenvalue problem for the *pair* of spatial Floquet exponents (γ_1, γ_2),

$$\mathbf{L}(x, z; \gamma_1, \gamma_2) \cdot \begin{pmatrix} \hat{\xi} \\ \hat{\eta} \\ \zeta \end{pmatrix} = 0. \tag{3.50}$$

Equation (3.50) is again a nonlinear in the parameter eigenvalue problem in *both* γ_1 and γ_2 and the degree is *four*. When the basic 3D state is known only approximately (as say a Fourier-Chebyshev series) the eigenvalue problem (3.50) will require significant computational effort. However, the simpler question of the role of spanwise bifurcations can be addressed by setting $\gamma_1 = 0$. Then (3.50) is an eigenvalue problem in one parameter, the spanwise Floquet exponent γ_2. Given γ_2 as a solution of (3.50) with $\gamma_1 = 0$ the spatial Floquet multiplier is $\exp[2\pi\gamma_2/\beta]$ and all the usual bifurcations (spanwise wavelength doubling, spanwise quasi-periodicity, etc.) are to be expected.

If the 3D basic doubly periodic state is of the standing variety, then $\mathbf{Z}_2^s$ is in its isotropy subgroup. In other words it is a reversible state (invariant under $z \mapsto -z$). In this case the spanwise tertiary bifurcations will be of the type found in reversible systems. The Floquet theory for reversible systems (with a reversible periodic orbit) is similar to Hamiltonian systems: periodic orbits are surrounded by tori generically and n-tupling bifurcations ($n \geq 3$) are of codimension 1 (rather than of codimension 2 as in non-reversible systems). Consequently the bifurcation structure of the spanwise tertiary bifurcations will differ from the bifurcations in streamwise direction. A spanwise Poincaré section will have a structure reminiscent of a symplectic map!

3.5 Spatial evolution of the primitive variables in 3D

The evolution equation for the primitive variables introduced in Section 2.3 is easily extended to the three-dimensional Navier-Stokes equations. Taking the divergence of the momentum equations in (3.1) results in the following Poisson equation for the pressure,

$$\Delta p + 2U_y v_x + u_x^2 + v_y^2 + w_z^2 + 2(v_x u_y + w_x u_z + w_y v_z) = 0. \tag{3.51}$$

The idea is to write the Poisson equation for p and the spanwise and vertical momentum equations as evolution equations and use the streamwise momentum equation as a constraint. Let

$$\Phi = \begin{pmatrix} v \\ V \\ w \\ W \\ p \\ q \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} v \\ v_x \\ w \\ w_x \\ p \\ p_x \end{pmatrix} \tag{3.52}$$

then we find

$$\frac{\partial}{\partial x}\Phi = \mathbf{L}(c, R) \cdot \Phi + \mathbf{N}(\Phi, u; R) \tag{3.53}$$

where

$$\mathbf{L}(c,R) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) & R(U-c) & 0 & 0 & R\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) & R(U-c) & R\frac{\partial}{\partial z} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2U_y & 0 & 0 & -(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}) & 0 \end{pmatrix} \tag{3.54}$$

and

$$\mathbf{N}(\Phi, u; R) = \begin{pmatrix} 0 \\ R(uV + vv_y + wv_z) \\ 0 \\ R(uW + vw_y + ww_z) \\ 0 \\ -2(v_y^2 + w_z^2 + v_y w_z + Vu_y + Wu_z + w_y v_z) \end{pmatrix} \tag{3.55}$$

and the constraint induced by the streamwise momentum equation is

$$\begin{aligned} q + \frac{1}{R}(V_y + W_z) - \frac{1}{R}(u_{yy} + u_{zz}) + U_y v - (U-c)(v_y + w_z) \\ + vu_y - uv_y + wu_z - uw_z = 0. \end{aligned} \tag{3.56}$$

Assuming the functions (u, v, w, p) have periodic spanwise variation, the evolution equation (3.53) is an $\mathbf{O}(2)$-equivariant vectorfield in a suitable function space.

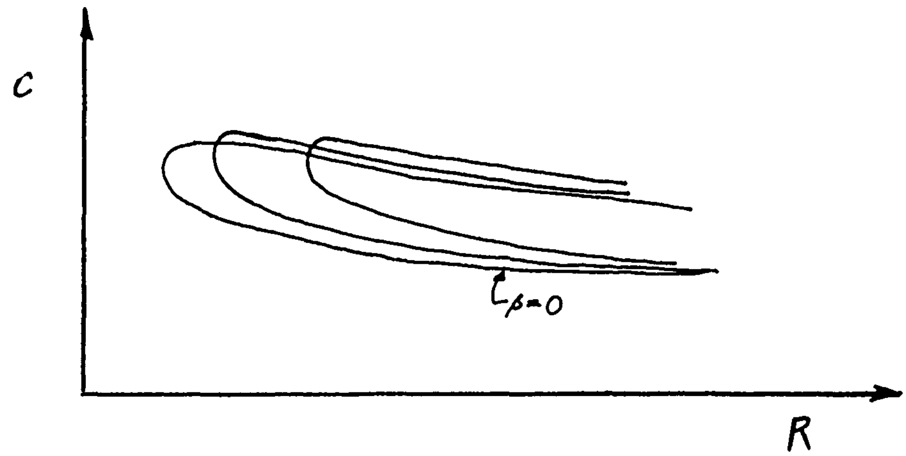

Figure 3.1 Neutral curves in the (c, R) plane for the modified (real) 3D Orr-Sommerfeld equation (3.7) for $\beta \geq 0$.

4. Wave interactions and spatially quasi-periodic states

Our claim is that along a branch of 2D spatially periodic states secondary bifurcation to spatial states that are quasi-periodic in the streamwise direction (and periodic in the spanwise direction) are to be expected. In this section a theory is presented that shows that *along the upper branch of the* 2D *neutral curve a secondary bifurcation to spatially quasi-periodic states is generic* (in the one-parameter family of 2D spatially periodic states). This is shown by analyzing the codimension 2 singularity associated with the intersection of the $\beta = 0$ and $\beta \neq 0$ neutral curves; in particular, the codimension 2 singularity brings the secondary bifurcation to quasi-periodic states down to the origin. Analysis of the unfolding of the singularity shows that secondary bifurcation to quasi-periodic states is codimension 1 along a branch of 2D spatially periodic states.

Suppose β is chosen arbitrarily in the interval $(0, \beta_{max})$ where $\beta_{max} \approx .3$ (for the Blasius boundary layer). The neutral curve for $\beta = 0$ and $\beta \neq 0$ is shown in Figure 4.1. In the two-parameter family (c, R) there is a codimension 2 point where the two neutral curves intersect. At the point (c_o, R_o) the 2D state and the 3D state will have pure imaginary eigenvalues; that is, at (c_o, R_o) the Orr-Sommerfeld equation (3.7) will have an eigenvalue $\lambda = i\alpha_1$ ($\alpha_1 \in \mathbb{R}$) when $\beta = 0$ and an eigenvalue $\lambda = i\alpha_2$ ($\alpha_2 \in \mathbb{R}$) when $\beta \neq 0$. Note that for each $\beta \in (0, \beta_{max})$ there exists a codimension 2 point, therefore such a codimension 2 point can be found at each point on the upper branch of the 2D neutral curve. For general $\beta \in (0, \beta_{max})$ the ratio α_1/α_2 will be *irrational* but at particular values of β the ratio will be rational. We suppose henceforth that the ratio α_1/α_2 is irrational and then treat the codimension 3 points (c_o, R_o, β_o) where $\alpha_1/\alpha_2 \in \mathbb{Q}$ as special cases.

As in Section 3.1 the $\mathbf{O}(2)$ symmetry forces the eigenvalue $\lambda = i\alpha_2$ (when $\beta \neq 0$) to be double. Therefore the spatial centre-manifold associated with the codimension 2 point (c_o, R_o) is six-dimensional. What we will show is that the codimension 2 point (c_o, R_o) can be treated as an $\mathbf{O}(2)$ equivariant (spatial) Hopf-Hopf mode-interaction on six-dimensions. The correspondence is useful because there is an interesting normal form theory due to Chossat, Golubitsky & Keyfitz [1986] (hereafter CGK) that is applicable. The theory of CGK is a temporal theory but nevertheless their existence results will be applicable here but the stability of the bifurcating quasi-periodic states will have to be determined by other methods.

4.1 Bifurcation of spatially quasi-periodic states

The evolution equation in the primitive variables given in Section 3.5 can be recast as an $\mathbf{O}(2)$-equivariant vectorfield in the following way. Assuming periodicity (wavenumber β) in the spanwise direction we can write

$$\Phi(x,y,z) = \tfrac{1}{2}\Phi_0(x,y) + \sum_{m=1}^{\infty} \Phi_{2m-1}(x,y)\cos m\beta z + \Phi_{2m}(x,y)\sin m\beta z \tag{4.1}$$

where

$$\Phi_j(x,y) = \begin{pmatrix} v_j(x,y) \\ V_j(x,y) \\ w_j(x,y) \\ W_j(x,y) \\ p_j(x,y) \\ q_j(x,y) \end{pmatrix} \qquad \text{with} \quad w_0 = W_0 = 0. \tag{4.2}$$

The point about the (spatial) Hopf-Hopf interaction can be made using the linear part of the evolution equation (in (3.53)-(3.54)). Let

$$\mathbf{L}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\partial^2}{\partial y^2} & R(U-c) & 0 & 0 & R\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2U_y & 0 & 0 & -\frac{\partial^2}{\partial y^2} & 0 \end{bmatrix}, \tag{4.3}$$

$$\mathbf{L}_m^+ = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ m^2\beta^2 - \frac{\partial^2}{\partial y^2} & R(U-c) & 0 & 0 & R\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & m^2\beta^2 - \frac{\partial^2}{\partial y^2} & R(U-c) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2U_y & 0 & 0 & m^2\beta^2 - \frac{\partial^2}{\partial y^2} & 0 \end{bmatrix} \tag{4.4}$$

and

$$\mathbf{L}_m^- = m\beta R\mathbf{E}_{45} \tag{4.5}$$

where $\mathbf{E}_{45}$ is a 6×6 matrix with unity at entry $(4,5)$ and zero everywhere else. With the matrices $\mathbf{L}_0$ and $\mathbf{L}_m^{\pm}$ the linear vectorfield $\frac{\partial}{\partial x}\Phi = \mathbf{L}(c,R)\Phi$ can be written

$$\frac{\partial}{\partial x}\begin{Bmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \vdots \\ \vdots \end{Bmatrix} = \begin{bmatrix} \mathbf{L}_0 & 0 & 0 & \cdots & \cdots & & \\ 0 & \mathbf{L}_1^+ & \mathbf{L}_1^- & 0 & 0 & & \\ 0 & -\mathbf{L}_1^- & \mathbf{L}_1^+ & 0 & 0 & & \\ \vdots & 0 & 0 & \mathbf{L}_2^+ & \mathbf{L}_2^- & 0 & 0 \\ \vdots & 0 & 0 & -\mathbf{L}_2^- & \mathbf{L}_2^+ & 0 & 0 \\ & & & 0 & 0 & \ddots & \\ & & & 0 & 0 & & \ddots \end{bmatrix} \begin{Bmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \vdots \\ \vdots \end{Bmatrix}. \tag{4.6}$$

Let $R_{m\theta} = \begin{pmatrix} \cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta \end{pmatrix}$ $m \in \mathbb{N}$ with $\theta \in \mathbb{R}$ be the usual rotation matrix on $\mathbb{R}^2$. Then an action for $\mathbf{O}(2)$ on the Fourier coefficient space is generated by

$$\mathbf{O}(2) = \langle \mathcal{R}, \mathcal{K} \rangle \tag{4.7}$$

where $\mathcal{R}$ generates $\mathbf{SO}(2)$ and is given by

$$\mathcal{R} = \mathrm{diag}(\mathbf{I}_6, R_\theta \otimes \mathbf{I}_6, R_{2\theta} \otimes \mathbf{I}_6, \ldots, R_{m\theta} \otimes \mathbf{I}_6, \ldots)$$

and $\mathcal{K}$ generates $\mathbf{Z}_2^\kappa$ and is given by

$$\mathcal{K} = \mathrm{diag}(\mathbf{I}_6, \kappa \otimes \mathbf{I}_6, \kappa \otimes \mathbf{I}_6, \ldots)$$

with $\kappa = \mathrm{diag}(1, -1)$. With the action of $\mathbf{O}(2)$ given in (4.7) it is clear that (4.6) is an $\mathbf{O}(2)$-equivariant vectorfield with $\mathbf{O}(2)$ acting trivially on the 2D state Φ_0. To study the spatial eigenvalue problem take $\Phi_j(x,y) = e^{\lambda x}\hat{\Phi}_j(y)$ then (4.6) decouples into the sequence of eigenvalue problems

$$\lambda\hat{\Phi}_0 = \mathbf{L}_0\hat{\Phi}_0 \tag{4.8}$$

$$\lambda\begin{pmatrix} \hat{\Phi}_{2m-1} \\ \hat{\Phi}_{2m} \end{pmatrix} = \begin{bmatrix} \mathbf{L}_m^+ & \mathbf{L}_m^- \\ -\mathbf{L}_m^- & \mathbf{L}_m^+ \end{bmatrix}\begin{pmatrix} \hat{\Phi}_{2m-1} \\ \hat{\Phi}_{2m} \end{pmatrix} \qquad m = 1, 2, \ldots. \tag{4.9}$$

Suppose $\lambda \in \sigma(\mathbf{L}_0)$ is a spatial Hopf bifurcation point ($\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\lambda) \neq 0$). A (spatial) Hopf-Hopf mode-interaction between a 2D state and a 3D state (equivalently a Hopf-Hopf mode interaction with $\mathbf{O}(2)$ symmetry on 6-dimensions) takes place if (for the same (c, R))

$$\lambda \in \sigma\begin{pmatrix} \mathbf{L}_m^+ & \mathbf{L}_m^- \\ -\mathbf{L}_m^- & \mathbf{L}_m^+ \end{pmatrix} \tag{4.10}$$

is also a spatial Hopf bifurcation point for some m. Without loss of generality we can take $m = 1$. The sequence of eigenvalue problems (4.9) is equivalent to (using (4.4) and (4.5))

$$\hat{v}''_{2m} + (\lambda^2 - m^2\beta^2)\hat{v}_{2m} = \lambda R(U-c)\hat{v}_{2m} + R\hat{p}'_{2m} \tag{4.11}$$
$$\hat{p}''_{2m} + (\lambda^2 - m^2\beta^2)\hat{p}_{2m} = -2\lambda U_y \hat{v}_{2m}$$
$$\hat{w}''_{2m} + (\lambda^2 - m^2\beta^2)\hat{w}_{2m} = \lambda R(U-c)\hat{w}_{2m} + m\beta R\hat{p}_{2m}, \tag{4.12}$$

with $\hat{\Phi}_{2m-1}$ satisfying the same set of equations (i.e. every eigenvalue of (4.11)-(4.12) is double). Clearly, a mode-interaction takes place (of six-dimensions) if there exists (c_o, R_o) at which (4.11) has a purely imaginary eigenvalue for both $m = 0$ and $m = 1$. This is in fact the case as shown in Figure 4.1; that is, a sufficient condition is that the neutral curves for $m = 0$ and $m = 1$ have a point of intersection.

For the bifurcation at the points (c_o, R_o) where a 2D and 3D state interact the normal form theory will be sketched with complete details along with numerical evaluation of the coefficients to be reported in Bridges [1991b]. At the critical point (c_o, R_o) suppose that $\lambda = i\alpha_1$ ($\alpha_1 \in \mathbb{R}$) when $\beta = 0$ and $\lambda = i\alpha_2$ ($\alpha_2 \in \mathbb{R}$) when $\beta \neq 0$. Then the linear solution in terms of the primitive variables at the point (c_o, R_o) is given by

$$\left.\begin{aligned}
\begin{pmatrix} u_1(x,y,z) \\ v_1(x,y,z) \\ p_1(x,y,z) \end{pmatrix} &= 2\mathrm{Re}\left[z_0 e^{i\alpha_1 x} \begin{pmatrix} \hat{u}_{11}(y) \\ \hat{v}_{11}(y) \\ \hat{p}_{11}(y) \end{pmatrix} \right] \\
&\quad + 2\mathrm{Re}\left[(z_1 e^{i(\alpha_2 x + \beta z)} + z_2 e^{i(\alpha_2 x - \beta z)}) \begin{pmatrix} \hat{u}_{12}(y) \\ \hat{v}_{12}(y) \\ \hat{p}_{12}(y) \end{pmatrix} \right] \\
\text{and} \qquad & \\
w_1(x,y,z) &= 2\mathrm{Re}\left[(z_1 e^{i(\alpha_2 x + \beta z)} - z_2 e^{i(\alpha_2 x - \beta z)})\hat{w}_{12}(y) \right]
\end{aligned}\right\} \tag{4.13}$$

where $(z_0, z_1, z_2) \in \mathbb{C}^3$ are complex amplitudes. Formal application of centre-manifold theory allows reduction to a vectorfield on $\mathbb{C}^3$. For the normal form Proposition 2.3 of CGK is adapted to the spatial setting. At the point (c_o, R_o) the normal form is

$$\frac{d}{dx}\begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = (p_0 + iq_0)\begin{pmatrix} z_0 \\ 0 \\ 0 \end{pmatrix} + (p_1 + iq_1)\begin{pmatrix} 0 \\ z_1 \\ z_2 \end{pmatrix} + (p_2 + iq_2)\delta\begin{pmatrix} 0 \\ z_1 \\ -z_2 \end{pmatrix} \tag{4.14}$$

where p_0, p_1, p_2, q_0, q_1 and q_2 are *real* functions of ρ, N and Δ where $\rho = |z_0|^2$, $N = |z_1|^2 + |z_2|^2$, $\Delta = \delta^2$ and $\delta = |z_2|^2 - |z_1|^2$. Moreover $q_0(0,0,0) = \alpha_1$ and $q_1(0,0,0) = \alpha_2$. An interesting property of the complex equations (4.14) is that the amplitudes and phases separate: let $z_j = r_j e^{i\phi_j}$ $j = 0,1,2$ then (4.14) decouples into

$$\left.\begin{aligned} \dot{r}_0 &= p_0(\rho, N, \Delta) r_0 \\ \dot{r}_1 &= (p_1(\rho, N, \Delta) + \delta p_2(\rho, N, \Delta)) r_1 \\ \dot{r}_2 &= (p_1(\rho, N, \Delta) - \delta p_2(\rho, N, \Delta)) r_2 \end{aligned}\right\} \tag{4.15}$$

and

$$\left.\begin{aligned} \dot{\phi}_0 &= q_0(\rho, N, \Delta) \\ \dot{\phi}_1 &= q_1(\rho, N, \Delta) + \delta q_2 \\ \dot{\phi}_2 &= q_1(\rho, N, \Delta) - \delta q_2. \end{aligned}\right\} \tag{4.16}$$

The set of amplitude equations in (4.15) is particularily easy to analyze because it is a $\mathbf{Z}_2 \oplus \mathbf{D}_4$ equivariant vectorfield. The group $\mathbf{Z}_2 = \langle F_0 \rangle$ and $\mathbf{D}_4 = \langle F_1, F_2, F \rangle$ where

$$\begin{aligned} F_0 \cdot (r_0, r_1, r_2) &= (-r_0, r_1, r_2) \\ F_1 \cdot (r_0, r_1, r_2) &= (r_0, -r_1, -r_2) \\ F_2 \cdot (r_0, r_1, r_2) &= (r_0, r_1, -r_2) \\ F \cdot (r_0, r_1, r_2) &= (r_0, r_2, r_1). \end{aligned} \tag{4.17}$$

CGK have used group-theoretic techniques to show the existence of seven classes of solutions in the normal form (4.15). They are listed in Table 4.1 along with their symmetry group (as subgroups of $\mathbf{Z}_2 \oplus \mathbf{D}_4$). Types 1,2 and 3 are the strictly periodic states that occur away from the interaction. States 5 and 7 require an additional parameter (are codimension 3). The interesting states are 4 and 6. They correspond to two classes of *spatially (streamwise) quasi-periodic states* with spanwise periodicity. Type 4 involves interaction between an oblique travelling wave and a 2D wave with intependent wavenumbers (*but the same phase speed*) and type 6 involves interaction between a standing 3D wave (actually travelling in the streamwise direction but periodic in the spanwise direction) and a 2D wave.

Further information about the bifurcating states can be obtained from the bifurcation equations. Expansion of the right hand side of (4.15) in a Taylor series and truncation at order 3 results in

$$
\begin{aligned}
p_0 &= c_0\lambda + \mu + a_0\rho + b_0 N \\
p_1 &= c_1\lambda + a_1\rho + b_1 N \qquad\qquad (4.18)\\
p_2 &= p_2^o
\end{aligned}
$$

where $c_0, a_0, b_0, c_1, a_1, b_1$ and p_2^o are real constants to be determined and (λ, μ) are the unfolding parameters of the codimension 2 singularity. CGK give a partial analysis of the bifurcation equations (4.18). There are numerous bifurcation sequences depending on the value of the coefficients. Computation of the coefficients relevant to the upper branch of the Blasius solution neutral curve are carried out in Bridges [1991b].

Ultimately the importance of the spatially quasi-periodic states that bifurcate along the upper branch of the 2D neutral curve will depend on whether they are stable or not. To determine stability of the spatial states will require reintroduction of time. For the spatially periodic states the sideband instability will have to be considered as well. More generally the spatial states correspond to spatial invariant manifolds. Therefore there will be two steps in the stability analysis: stability with respect to parametrically equivalent manifolds (i.e. in the spatially periodic case, stability with respect to perturbations of the same wavenumber) and secondly stability with respect to other "nearby" spatial invariant manifolds (this is a generalization of the sideband instability). A stability theory for spatially periodic state including sideband instability is straightforward but a theory for the stability of the more complex spatial invariant manifolds is by no means clear but is clearly of great importance for determining the stable spatial states in shear flows.

4.2 Spanwise resonances and mode-interactions on 8-dimensions

Although we do not pursue it here it is possible to have interactions between two 3D states resulting in an eight-dimensional centre-manifold. Maintaining the basic assumption of periodicity in the spanwise direction we look for resonances between two

3D blocks ($m > 0$) in (4.9). For example consider block $m = 1$ and $m = 2$ which would correspond to a *spanwise* resonance between β and 2β. Figure 4.2 shows an example of neutral curves in the (c, R) plane for β and 2β (obtained by solving the eigenvalue problem (4.11)). At the point of intersection of the two curves both β and 2β result in purely imaginary eigenvalues of (4.11); that is, $\lambda = i\alpha_1$ ($\alpha_1 \in \mathbb{R}$) corresponds to β (at fixed (c_o, R_o)) and 2β results in $\lambda = i\alpha_2$ ($\alpha_2 \in \mathbb{R}$). Each eigenvalue is double due to the O(2) symmetry (and α_1/α_2 is generically irrational) resulting in an 8-dimensional centre-manifold, This interaction corresponds to a Hopf-Hopf mode-interaction with O(2) symmetry on 8-dimensions which has been studied by Chossat, Golubitsky & Keyfitz [1986]. The interaction is very complex and produces quasi-periodic solutions (in the present case *spatially* quasi-periodic states) with 2, 3 *and* 4 independent frequencies (or wavenumbers in the present case). The 8-dimensional mode-interaction will occur at higher Reynold's number than the 6-dimensional mode-interaction and therefore it would appear to be of less importance. However, there is an interesting dynamical feature in the eight-dimensiona interaction. Melbourne, Chossat & Golubitsky [1988] have shown that *heteroclinic cycles* can be found in mode-interactions on 8-dimensions. Aubry, et al. [1988] have introduced a model for *fully developed turbulence* in boundary layers which shows that *heteroclinic cylcles* provide a good theoretical model for the bursting phenomena. Therefore analysis of the 8-dimensional interaction in *transitional* boundary layers may provide a prelude to "dynamical" behavior that persists in fully developed turbulent boundary layers.

4.3 Resonant triads

In the codimension two non-resonant interaction treated in Section 4.1 it was noted that at particular values of β the interaction is resonant; that is, there exists distinguished points (c_o, R_o, β_o) at which the 2D state has wavenumber α_1, the 3D state has spanwise wavenumber β_o and wavenumber $p\alpha_1/q$ where (p, q) are integers. The interesting resonances are when $p = 1$ and $q = 2,3$ or 4 (strong resonances). It is easy to show that each of the strong resonances occur in the Blasius boundary layer. Indeed, the $(p,q) = (1,2)$ resonance corresponds to the Craik resonant triad although our nonlinear theory will differ: Craik treats the wave speed as complex and shows that the energy in an unstable 2D wave is transferred to the 3D wave. In our theory the wavespeed is taken as *real* and the resonant interaction is treated as a *spatial* (codimension 2 Hopf) bifurcation.

The resonant interactions are demonstrated as follows. Using the (c,R) neutral curve, pick a value of c at which there are two R-intersections (R_0 and R_1) as shown in Figure 4.3. Corresponding to (c,R_0) is wavenumber α_0 and to (c,R_1) is wavenumber α_1. Now map $(\alpha_0,R_0) \to (\alpha_2,R_1,\beta)$ with $\beta \neq 0$ using the Squire transformation. Then

$$\alpha_2 R_1 = R_0\alpha_0 \qquad \alpha_2 = \sqrt{\alpha_0^2 - \beta^2}.$$

Therefore corresponding to Reynolds number R_1 there is a 2D wave with wavenumber α_1 and a 3D wave with wavenumber α_2. Write their ratio as $\rho = \alpha_1/\alpha_2$ then

$$\rho = \frac{\alpha_1 R_1}{\alpha_0 R_0}$$

which is easily constructed from the neutral curve data and is plotted as a function of c in Figure 4.4. A resonant interaction occurs whenever $\rho = \frac{p}{q}$. In general $\rho > 1$ but it is clear from Figure 4.4 that there exist values of (c,R,β) at which $\rho = 2,3$ and 4 but in general ρ will be irrational (corresponding to the states in Section 4.1).

The normal form for the resonant interactions $\rho = 2,3$ and 4 will be more difficult than the normal form for the non-resonant interaction (less symmetry). The normal form symmetry will be $\mathbf{O}(2)\times\mathbf{S}^1$ on 6-dimensions. Consequently the amplitude/phase equations do not separate. Normal forms for resonant Hopf-Hopf interactions with $\mathbf{O}(2)$ symmetry on 6-dimensions have not appeared in the literature. This is an interesting area for further research and will be of great interest for understanding the flow near the resonant points on the upper branch of the 2D neutral curve.

Table 4.1

Solution types in the unfolding of the codimension-2 singularity (c_o, R_o)

	Isotropy Subgroup Σ	Fix (Σ)	Vectorfield on Fix (Σ)	Solution Type
0	$\mathbf{Z}_2 \times \mathbf{D}_4$	0	–	trivial solution
1	$\{1\} \times \mathbf{D}_4$	$(r_0, 0, 0)$	$p_0 = 0$	2D spatially periodic state
2	$\mathbf{Z}_2 \times \{\mathsf{F}_2, 1\}$	$(0, r_1, 0)$	$p_1 - r_1^2 p_2 = 0$	3D (oblique) spatially periodic state
3	$\mathbf{Z}_2 \times \{\mathsf{F}, 1\}$	$(0, r_1, r_1)$	$p_1 = 0$	3D (standing) spatially periodic state
4	$\{1\} \times \{\mathsf{F}_2, 1\}$	$(r_0, r_1, 0)$	$p_0 = 0$ $p_1 - r_1^2 p_2 = 0$	2D-3D(oblique) quasi-periodic interaction
5	$\mathbf{Z}_2 \times \{1\}$	$(0, r_1, r_2)$	$p_1 = 0$ $p_2 = 0$	3D(standing)-3D(oblique) quasi-periodic interaction
6	$\{1\} \times \{\mathsf{F}, 1\}$	(r_0, r_1, r_1)	$p_0 = 0$ $p_1 = 0$	2D-3D(standing) quasi-periodic interaction
7	$\{1\}$	(r_0, r_1, r_2)	$p_0 = 0$ $p_1 = 0$ $p_2 = 0$	2D-3D(oblique)-3D(standing) quasi-periodic (3-torus) interaction

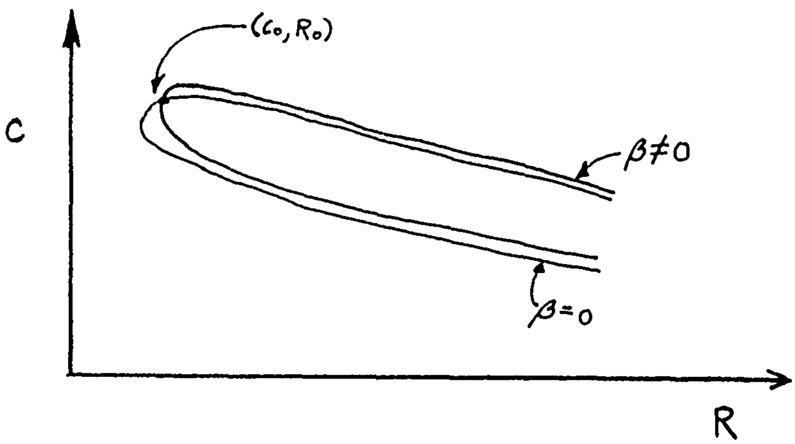

Figure 4.1 Neutral curve of the Orr-Sommerfeld equation for $\beta = 0$ and $\beta \neq 0$ illustrating the codimension 2 intersection point.

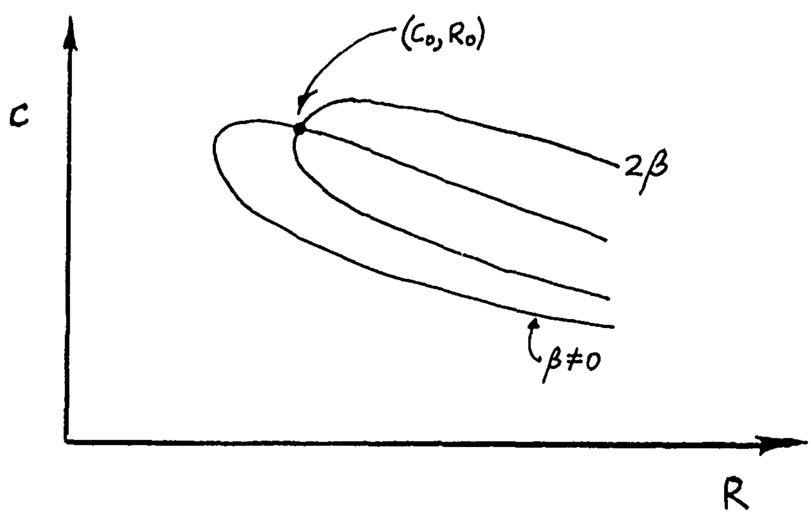

Figure 4.2 Neutral curve for β and 2β illustrating the interaction point for spanwise resonance.

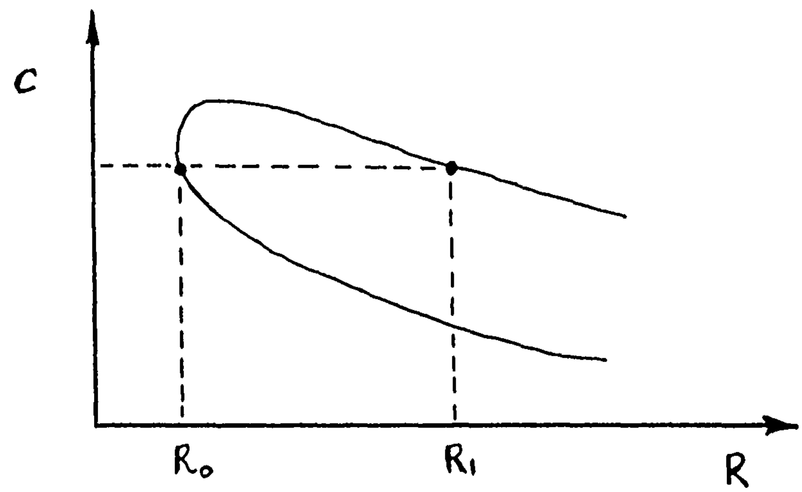

Figure 4.3 Finding resonant and non-resonant interaction points using Squire's theorem.

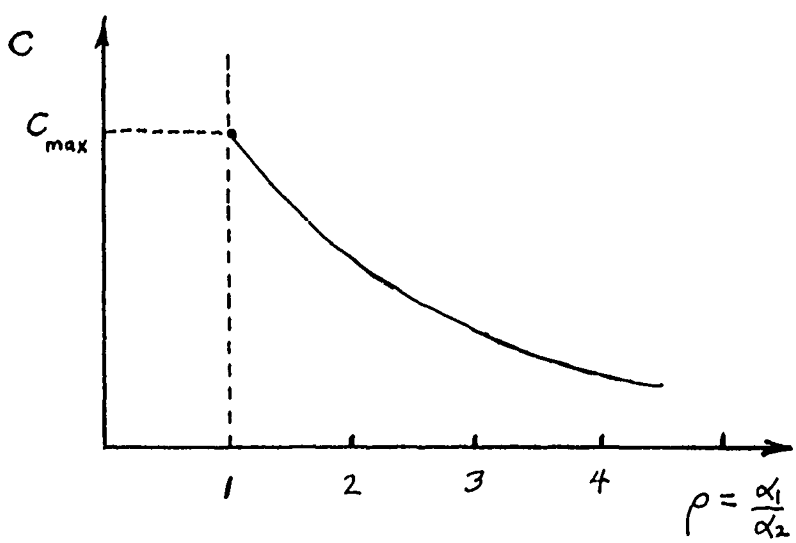

Figure 4.4 Ratio of the wavenumbers in the 2D-3D wave-interaction along the upper branch of the 2D neutral curve.

5. 2D spatially quasi-periodic states and the compliant wall problem

The introduction of a compliant wall to the boundary layer problem has a two-fold effect. Research of Carpenter & Garrad [1985], Carpenter & Morris [1990] and others has demonstrated that proper use of a compliant wall (i.e. optimal material properties, geometry and construction) leads to a reduction in drag and a greater control over the point of transition in the boundary layer. On the other hand the potential for "wall dynamics" leads to new instabilities and competing instabilities in the boundary layer. It is this latter feature that interests us here. A classification of the instabilities (and the nonlinear bifurcations associated with them) is of great practical importance for the design of a compliant wall (in other words they need to be understood so they can be avoided (or used to advantage!)).

There are a number of singularities in the linear analysis of the stability of the compliant wall problem (see Carpenter [1990]). Here we consider a particular singularity in the neutral curve for the compliant wall problem that supports our theory of secondary bifurcation to spatially quasi-periodic states.

In Section 4 the emphasis was on secondary bifurcation to spatially (streamwise) quasi-periodic state that have spanwise periodicity. In other words the streamwise quasi-periodicity appeared *with* three-dimensionality and the basic 2D state was spatially periodic. However, it was shown in Section 2.2 that secondary bifurcation *from* spatially periodic 2D states *to* spatially quasi-periodic 2D states was possible. Identification of the points of secondary bifurcation to quasi-periodic 2D states will require numerical calculation. An alternative is to use the "codimension-2 strategy". The secondary bifurcation to 2D quasi-periodic states is of codimension 1; that is, such bifurcations are generic in the one-parameter family of 2D spatially periodic states. In the codimension-2 strategy we introduce another parameter that brings the secondary bifurcation point down to the origin. In particular we will study a singularity in the neutral curve found by Carpenter & Garrad [1985, Figure 11] by varying a single parameter (the elastic modulus of the wall). What we intend to show is that the singularity found by Carpenter & Garrad is associated with a bifurcation to spatially quasi-periodic 2D states. This singularity in question may not be of great practical importance to the compliant wall problem (it appears at a Reynolds number of about 3000 (based on displacement thickness) whereas the initial 2D instability occurs at $R \approx 500$) but it is nevertheless significant in demonstrating that bifurcation to more complex 2D spatial structures (other than spatially periodic) is to be anticipated.

Consider the linear coupled fluid-wall problem. The fluid is governed by the linear 2D Navier-Stokes equations (in the convective frame) given in Equations (2.2). For a *rigid wall* the wall boundary conditions are $u = v = 0$ at $y = 0$. For a compliant wall the governing equation, when the wall is modeled as a simple beam, is

$$C_M^* \frac{\partial^2 \eta}{\partial t^2} + C_B^* \frac{\partial^4 \eta}{\partial x^4} + C_{KE}^* \eta = -p_w \tag{5.1}$$

where $\eta(x,t)$ is the vertical displacement of the wall and C_M^*, C_B^* and C_{KE}^* are dimensionless (using displacement thickness variables) coefficients and $p_w(x,0,t)$ is the fluid pressure at the wall. The linear kinematic conditions at the wall are given by

$$v = \frac{\partial \eta}{\partial t} \quad \text{and} \quad u = -U_y(0)\eta. \tag{5.2}$$

With the transformation $x \mapsto x - ct$ the equations (5.1) and (5.2) can be written as

$$\left.\begin{aligned} & c\frac{\partial^2 v}{\partial x \partial y} + U_y(0)\frac{\partial v}{\partial x} = 0, \qquad v + c\frac{\partial \eta}{\partial x} = 0 \\ & \text{and} \\ & C_B^* \frac{\partial^4 \eta}{\partial x^4} + c^2 C_M^* \frac{\partial^2 \eta}{\partial x^2} + C_{KE}^* \eta = -p_w. \end{aligned}\right\} \tag{5.3}$$

The linear Navier-Stokes equations (2.2) together with the boundary conditions (5.3) (and appropriate boundary conditions at infinity) are considered as an evolution equation in x. Taking the dynamical systems approach, let $(u, v, p, \eta) = e^{\lambda x}(\hat{u}, \hat{v}, \hat{p}, \hat{\eta})$ then the fluid equations reduce to the (modified (real)) Orr-Sommerfeld equation,

$$\left(\frac{d^2}{dy^2} + \lambda^2\right)^2 \hat{v} + \lambda R U_{yy} \hat{v} - \lambda R(U - c)\left(\frac{d^2}{dy^2} + \lambda^2\right)\hat{v} = 0 \tag{5.4}$$

and the boundary conditions (5.3) reduce to

$$\left.\begin{aligned} c\frac{d\hat{v}}{dy} + U_y(0)\hat{v} = 0 \\ \frac{d^3\hat{v}}{dy^3} + \lambda^2\frac{d\hat{v}}{dy} + \frac{\lambda}{c}\left(\frac{\lambda^4}{R}C_B + \lambda^2 c^2 C_M + C_{KE}R^2\right)\hat{v} = 0 \end{aligned}\right\} \text{ at } y = 0. \qquad (5.5)$$

Equations (5.4) and (5.5) together with appropriate boundary conditions at infinity ($\hat{v}, \hat{v}' \to 0$ as $y \to \infty$) form a *nonlinear in the parameter* eigenvalue problem (of degree 5) for λ as a function of c, R and the wall parameters C_B, C_M and C_{KE}. The variables C_B, C_M and C_{KE} are introduced to eliminate the streamwise dependence of C_B^*, C_M^* and C_{KE}^* (see Carpenter & Garrad [1985, eqn. (6.5)]),

$$C_B^* = C_B R^{-3}, \quad C_{KE}^* = C_{KE} R \quad \text{and} \quad C_M^* = C_M R^{-1}.$$

The boundary conditions (5.5) are in a form rather different from Carpenter & Garrad and Carpenter & Morris; here we suppose $c \in \mathbf{R}$ and $\lambda \in \mathbf{C}$ is a spatial eigenvalue. In Carpenter & Morris the classic spatial stability approach is used; that is, the *frequency* $\omega \in \mathbf{R}$, $c \in \mathbf{C}$ and α, the wavenumber, is the eigenvalue.

Figure 5.1 shows a schematic of the results of Carpenter & Garrad [1985] (taken from their Figure 11, p. 498). By varying the elastic modulus (C_M fixed and C_{KE} and C_B varying dependent on E) the neutral curve varies dramatically. In particular as the elastic wall modulus is decreased there exists a critical value of $E = E_o$ at which the neutral curve breaks into two pieces. Our interest is in the critical point $E = E_o$ when the upper and lower branches of the neutral curve first intersect. Carpenter & Garrad treated the problem from the temporal point of view (real wavenumber α with $c \in \mathbf{C}$ the eigenvalue) and plotted the neutral curve in $\alpha - R$ space. In Figure 5.2 the singularity is viewed in the (c, R) plane (E_o will differ slightly from the temporal value in Figure 5.1). In other words there exist $E = E_o$ at which the upper and lower branch of the neutral curve (in the (c, R) plane intersect. Recall that associated with each point along the neutral curve is an eigenvalue λ with $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\lambda) \neq 0$. Therefore when $E = E_o$ and $(c, R) = (c_o, R_o)$ the linear problem will have eigenvalues $\lambda = \pm i\alpha_1$ and $\lambda = \pm i\alpha_2$ with α_1/α_2 (generically) irrational; that is, *the linear problem has spatially* *quasi-periodic solutions* and a *center* subspace of 4 dimensions. The vertical velocity will have the form

$$v(x,y) = 2\mathrm{Re}\left[A\hat{v}_1(y)e^{i\alpha_1 x} + B\hat{v}_2(y)e^{i\alpha_2 x}\right] \tag{5.6}$$

where $A, B \in \mathbf{C}$ are complex amplitudes (note that $\hat{v}_1$ and $\hat{v}_2$ are in general distinct functions) and *the wall displacement will also be spatially quasi-periodic*,

$$\eta(x) = \frac{2}{c_o}\mathrm{Re}\left[\frac{i}{\alpha_1}A\hat{v}_1(0)e^{i\alpha_1 x} + \frac{i}{\alpha_2}B\hat{v}_2(0)e^{i\alpha_2 x}\right] \tag{5.7}$$

with related expressions for $u(x,y)$ and $p(x,y)$.

The idea is to apply centre-manifold theory to reduce the spatial evolution equation to a vectorfield on $\mathbf{R}^4$ to which normal form theory is applied to show the bifurcation of nonlinear spatially quasi-periodic states. The analysis is sketched here with complete details to found in Bridges [1991d]. Let $f = (\mathrm{Re}(A), \mathrm{Im}(A), \mathrm{Re}(B), \mathrm{Im}(B)) \in \mathbf{R}^4$ then the (formal) centre-manifold reduction (as in Section 3.1) can be used to construct a reduced vectorfield for f;

$$\frac{d}{dx}f = L(c_o, R_o)\cdot f + N(f, c_o, R_o) \qquad E = E_o \tag{5.7}$$

with

$$L(c_o, R_o) = \begin{bmatrix} 0 & \alpha_1 & & \\ -\alpha_1 & 0 & & \\ & & 0 & \alpha_2 \\ & & -\alpha_2 & 0 \end{bmatrix}.$$

The problem has been reduced to a vectorfield on $\mathbf{R}^4$ in which the linear part has 2 purely imaginary pairs of eigenvalues without resonance, a singularity that has been analyzed by Takens and by Guckenheimer & Holmes [1983, Section 7.5]. If we set $r_1 = |A|$ and $r_2 = |B|$ then successive changes of variables reduces (5.7) to a $\mathbf{T}^2$-equivariant normal form (to some order). To third order the normal form for the amplitudes reduces to

$$\begin{aligned} \frac{dr_1}{dx} &= r_1(\mu_1 + r_1^2 + br_2^2) \\ \frac{dr_2}{dx} &= r_2(\mu_2 + cr_1^2 + dr_2^2) \quad d = \pm 1 \end{aligned} \tag{5.8}$$

(Guckenheimer & Holmes equation (7.5.2)) where (μ_1, μ_2) are the two unfolding parameters (related to $c - c_o$ and $R - R_o$ in the present case). Numerical calculations of the coefficients b, c and d are carried out in Bridges [1991d] for the Blasius boundary layer adjacent to a compliant wall. Unfolding of the normal form (5.8) shows that the two branches of spatially periodic states will persist. In addition there are secondary bifurcations to quasi-periodic states and if certain parametric conditions are met there is a tertiary bifurcation to 3-tori. Schematic bifurcation diagrams are shown in Figures 5.3(a) and (b). In Figure 5.3(a) the secondary branch of quasi-periodic states goes off to infinity whereas in Figure 5.3(b) the secondary branch connects the two branches of periodic states and includes a tertiary bifurcation to a 3-torus. Note that stability assignments are not included in Figure 5.3. The centre-manifold amplitudes and the normal form in (5.8) are written in terms of *spatial* evolution so only existence results are obtained. Determination of the stability properties of the quasi-periodic states is a non-trivial problem and will require the reintroduction of time.

In obtaining the singularity in Figure 5.1 Carpenter & Garrad varied only one parameter. Note that even in the simple model of the compliant wall (5.1) there are three independent parameters (the more sophisticated models of Carpenter & Garrad and Carpenter & Morris contain considerably more parameters). Our claim is that *another parameter can be varied to bring the two non-resonant wavenumbers in the (spatial) Hopf bifurcation together* as shown in Figure 5.4. The configuration in Figure 5.4 is the (codimension 3) 1:1 nonsemisimple Hopf bifurcation and has been analyzed by van Gils, Krupa & Langford [1990]. This singularity is of interest for two reasons. From a practical point of view the location of high codimension singularities in the parameter space of the compliant wall problem is of interest in order to "design around them". From a theoretical point of view the 1:1 nonsemisimple (spatial) Hopf bifurcation introduces new spatial bifurcations; in particular, van Gils, Krupa & Langford show that the unfolding of the 1:1 nonsemisimple Hopf contains homoclinic bifurcations and period doubling bifurcations as well as n-tori ($n = 2$ and 3). Adaptation to the spatial setting will result in interesting spatial structures; in particular, the "spatial homoclinic" will correspond to a soliton-like feature in the shear flow!

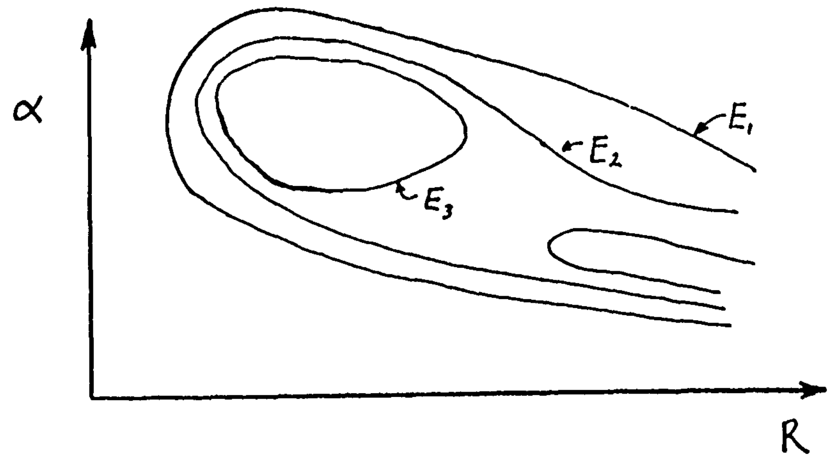

Figure 5.1 Effect of reducing the wall elastic modulus (E) on the neutral curve for the Blasius boundary layer (after Carpenter & Garrad [1985, Figure 11]).

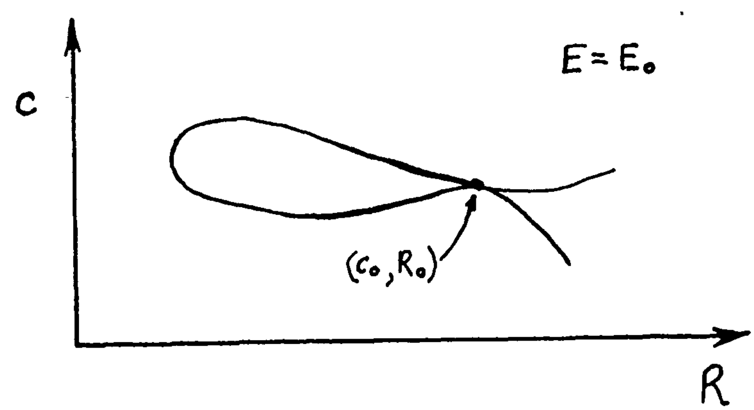

Figure 5.2 Neutral curve at the critical value of the wall elastic modulus $E = E_o$ in the (c, R) plane.

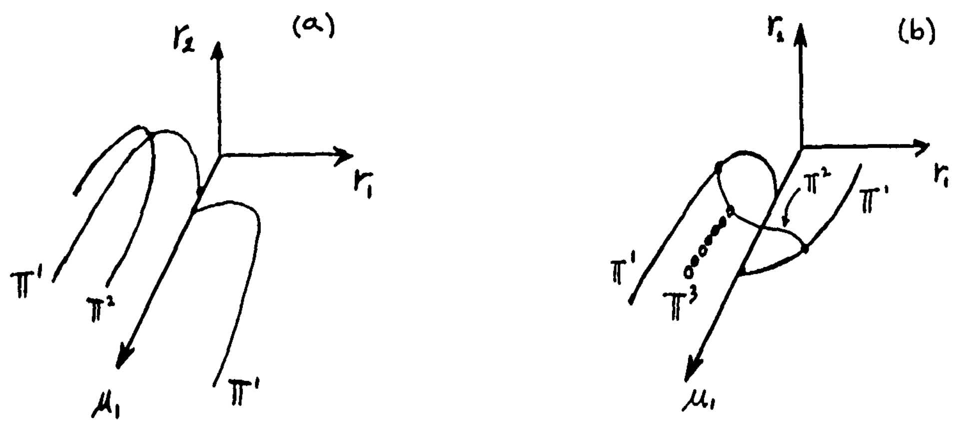

Figure 5.3 Schematic bifurcation diagrams for the normal form in equation (5.8) showing how secondary bifurcations to 2-tori and 3-tori arise: (a) infinite branch of T^2 and (b) finite secondary branch of T^2 with tertiary bifurcation to T^3.

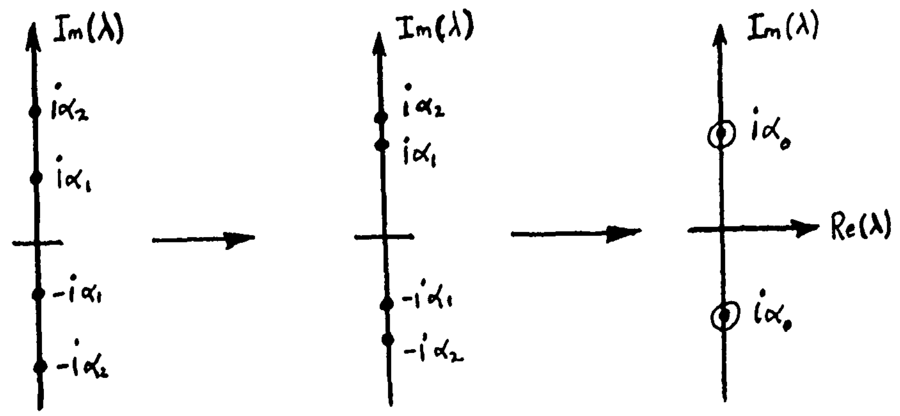

Figure 5.4 Coalescence of the non-resonant Hopf-Hopf interaction by the addition of a third parameter producing a 1 : 1 non-semisimple Hopf.

References

[1] N. AUBRY, P. J. HOLMES, J. L. LUMLEY & E. STONE [1988] *The dynamics of coherent structures in the wall region of a turbulent boundary layer*, J. Fluid Mech. **192**, pp. 115-173

[2] B. J. BAYLY, S. A. ORSZAG & T. HERBERT [1988] *Instability mechanisms in shear-flow transition*, Ann. Rev. Fluid Mech. **20**, pp. 359-91

[3] B. L. J. BRAAKSMA, H. W. BROER & G. B. HUITEMA [1990] *Toward a quasi-periodic bifurcation theory*, Memoirs AMS **83**, pp. 1-82

[4] T. J. BRIDGES [1991a] *A dynamical systems approach to transition in shear flows*, in preparation

[5] T. J. BRIDGES [1991b] O(2)-*equivariant spatial-Hopf bifurcation and quasi-periodic wave-interactions in the Blasius boundary layer*, in preparation

[6] T. J. BRIDGES [1991c] *Secondary bifurcation of spatially quasi-periodic states in channel flow*, in preparation

[7] T. J. BRIDGES [1991d] *Spatially quasiperiodic waves in the Blasius boundary layer adjacent to a compliant wall*, in preparation

[8] T. J. BRIDGES [1991e] *Degenerate period doubling and bubbling*, in preparation

[9] T. J. BRIDGES & P. J. MORRIS [1984a] *Differential eigenvalue problems in which the parameter appears nonlinearly*, J. Comp. Physics **55**, pp. 437-60

[10] T. J. BRIDGES & P. J. MORRIS [1984b] *Spectral calculations of the spatial stability of non-parallel boundary layers*, AIAA Paper No. 84-0437

[11] T. J. BRIDGES & P. J. MORRIS [1987] *Boundary layer stability calculations*, Physics of Fluids **30**(11), pp. 3351-8

[12] H. W. BROER, G. B. HUITEMA & F. TAKENS [1990] *Unfoldings of quasi-periodic tori*, Memoirs AMS **83**, pp. 83-175

[13] P. W. CARPENTER [1990] *Status of Transition Delay Using Compliant Walls*, Prog. in Astro. and Aero. **123**, pp. 79-113

[14] P. W. Carpenter & A. D. Garrad [1985] *The hydrodynamic stability of flow over Kramer-type compliant surfaces. Part 1. Tollmien- Schlichting instabilities*, J. Fluid Mech. **155**, pp. 465-510

[15] P. W. Carpenter & P. J. Morris [1990] *The effect of anisotropic wall compliance on boundary-layer stability and transition*, J. Fluid Mech. **218**, pp. 171-223

[16] P. Chossat & M. Golubitsky [1988] *Iterates of maps with symmetry*, SIAM J. Math. Anal. **19**, pp. 1259-1270

[17] P. Chossat, M. Golubitsky & B. Keyfitz [1986] *Hopf-Hopf mode interations with* O(2) *symmetry*, Dyn. Stab. Sys. **1**, pp. 255-292

[18] P. H. Coullet & E. A. Spiegel [1983] *Amplitude equations for systems with competing instabilities* SIAM J. Appl. Math. **43**, pp. 776-821

[19] A. D. D. Craik [1971] *Nonlinear resonant instability in boundary layers*, J. Fluid Mech. **50**, pp. 393-413

[20] A. D. D. Craik [1985] *Wave interactions and fluid flows*, Cambridge University Press

[21] P. G. Drazin & W. Reid [1981] *Hydrodynamic Stability*, Cambridge University Press

[22] M. S. P. Eastham [1973] *The Spectral Theory of Periodic Differential Equations*, Scottish Academic Press, Edinburgh

[23] C. Elachi [1976] *Waves in active and passive periodic structures: a review*, Proc. IEEE **64**, pp. 1666-1698

[24] M. Gaster [1962] *A note on the relation between temporally increasing and spatially increasing disturbances in hydrodynamic stability*, J. Fluid Mech. **14**, pp. 222-4

[25] M. Gaster [1990] *On the nonlinear phase of wave growth leading to chaos and breakdown to turbulence in a boundary layer as an example of an open system*, Proc. Roy. Soc. **A430**, pp. 3-24

[26] S. A. van Gils, M. Krupa & W. .F. Langford [1990] *Hopf bifurcation with non-semisimple 1:1 resonance*, Nonlinearity **3**, pp. 825-850

[27] M. GOLUBITSKY & M. ROBERTS [1987] *A classification of degenerate Hopf bifurcations with O(2) symmetry*, J. Diff. Eqns. **69**, pp. 216-64

[28] M. GOLUBITSKY, I. STEWART & D. SCHAEFFER [1988] *Singularities and groups in bifurcation theory, Vol. II, Appl. Math. Sci. no. 69*, Springer-Verlag

[29] C. GROSCH & H. SALWEN [1978] *The continuous spectrum of the Orr-Sommerfeld problem. Part 1. The spectrum and the eigenfunctions*, J. Fluid Mech. **87**, pp. 33-54

[30] J. GUCKENHEIMER & P. J. HOLMES [1983] *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Appl. Math. Sci. **42**, Springer-Verlag

[31] T. HERBERT [1975] *On finite amplitudes of periodic disturbances on the boundary layer along a flat plate*, Proc. 4^{th} Int. Conf. Numer. Meth. Fluid Dyn. **LNP-35**, pp. 212-7

[32] T. HERBERT [1983] *Secondary instability of plane channel flow to subharmonic three-dimensional disturbances*, Physics of Fluids **26**, pp. 871-4

[33] T. HERBERT [1984] *Analysis of the subharmonic route to transition in boundary layers*, AIAA Paper No. 84-0009

[34] T. HERBERT [1988] *Secondary instability of boundary layers*, Ann. Rev. Fluid Mech. **20**, pp. 487-526

[35] YU. S. KACHANOV & V. YA. LEVCHENKO [1984] *The resonant interaction of disturbances at laminar-trubulent transition in a boundary layer*, J. Fluid Mech. **138**, pp. 209-247

[36] K. KIRCHGÄSSNER [1982] *Wave solutions of reversible systems and applications*, J. Diff. Eqns. **45**, pp. 113-27

[37] I. MELBOURNE, P. CHOSSAT & M. GOLUBITSKY [1988] *Heteroclinic cycles involving periodic solutions in mode interactions with* O(2) *symmetry*, Research report UH/MD-47, University of Houston

[38] S. A. ORSZAG & A. T. PATERA [1983] *Secondary instability of wall-bounded shear flows*, J. Fluid Mech. **128**, pp. 347-85

[39] A. J. PEARLSTEIN & D. A. GOUSSIS [1988] *Efficient transformation of certain singular polynomial matrix eigenvalue problems*, J. Comp. Physics **78**, pp. 305-12

[40] F. T. SMITH [1979] *On the non-parallel flow stability of the Blasius boundary layer*, Proc. Royal Soc. London **A366**, pp. 91-109

[41] A. VANDERBAUWHEDE [1989] *Period-doublings and orbit-bifurcations in symmetric systems*, Dyn. Sys. & Ergodic Theory **23**, Banach Center Publications, pp. 197-208

[42] A. VANDERBAUWHEDE [1990] *Subharmonic bifurcation in equivariant systems*, preprint, Rijksuniversiteit Gent, Belgium

[43] A. VANDERBAUWHEDE & G. IOOSS [1990] *Center manifold theory in infinite dimensions*, preprint, Université de Nice

[44] J. WATSON [1962] *On spatially-growing finite disturbances in plane Poiseuille flow*, J. Fluid Mech. **14**, pp. 211-21

[45] A. A. WRAY & M. Y. HUSSAINI [1984] *Numerical experiments in boundary layer stability*, Proc. Royal Soc. London **A392**, pp. 373-89