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Abstract

Recent theoretical work by Hall & Seddougui (1989) has shown that strongly non-linear, high wavenumber Görtler vortices developing within a boundary layer flow are susceptible to a secondary instability which takes the form of travelling waves confined to a thin region centred at the outer edge of the vortex. This work considered the case in which the secondary mode could be satisfactorily described by a linear stability theory and in the current paper our objective is to extend this investigation of Hall & Seddougui (1989) into the nonlinear regime. We find that at this stage not only does the secondary mode become nonlinear but it also interacts with itself so as to modify the governing equations for the primary Görtler vortex. In this case then, the vortex and the travelling wave drive each other and, indeed, the whole flow structure is described by an infinite set of coupled, nonlinear differential equations. We undertake a Stuart–Watson type of weakly nonlinear analysis of these equations and conclude, in particular, that on this basis there exist stable flow configurations in which the travelling mode is of finite amplitude. Implications of our findings for practical situations are discussed and it is shown that the theoretical conclusions drawn here are in good qualitative agreement with available experimental observations.

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§1. Introduction.

Our concern is with an asymptotic description of the three-dimensional breakdown of steady, spanwise periodic, large Görtler vortices via nonlinear travelling waves and is a natural extension of the work of Hall & Seddougui (1989) who investigated the current problem in the context of breakdown through very small waves.

Experiments by Bippes & Görtler (1972) and Aihara & Koyama (1981) suggest that the breakdown of vortices leads to a flow with wavy boundaries and which is periodic in time. Later, Kohama (1987) studied vortex instabilities in boundary layers over a laminar flow control wing, and observed that a secondary instability of Görtler vortices appeared which was confined to a thin zone at the top of the extent of vortex activity. Additionally, he noted that the instability propagated with a speed that approached the free stream speed as the disturbance travelled downstream, and the onset of the unsteady flow perturbations was eventually followed by transition to turbulence. Peerhossaini & Wesfreid (1988) studied the fluid flow in a boundary layer in a concave section of a curved channel. Like Kohama, they found that when the secondary breakdown of Görtler vortices appeared it did so at the top of the vortices. Further downstream they reported another instability, but which was confined to a zone close to the wall.

Much of the early theoretical work concerned with Görtler vortices concentrated on the linear stability of external flows over concave walls. Notable contributions were made by, among others, Görtler (1940), Smith (1955), Hämmerlin (1956) and Floryan & Saric (1979). However, Hall (1982a, b, 1983) demonstrated that much of this early work was fundamentally flawed for all the previous analyses had invoked the parallel flow approximation (which essentially assumes that the basic flow in which the vortices lie is independent of the streamwise co-ordinate and so neglects the effect of boundary layer growth). This approximation enables the linear instability equations to be expressed as ordinary differential equations but Hall illustrated that this assumption is unjustifiable except in the limit of small vortex wavelength, and indeed this is the explanation for the considerable inconsistencies in the results of the previous studies. Additionally, in the case of small vortex wavelength the Görtler instability may be described by an asymptotic structure which takes account of boundary layer growth in a fully rational manner and thence the parallel flow approximation is rendered superfluous in the only situation in which it has any relevance whatsoever. For larger wavelengths the problem is fully non-parallel and the linear stability equations, which now take the form of a set of partial differential equations, have to be solved numerically, see Hall (1983). This paper showed two significant features of this non-parallel flow problem, namely that the ideas of a unique neutral stability curve and of unique growth rates at a specified downstream location are inapplicable to the Görtler problem because the location where a vortex commences to grow is dependent upon the position and the size and shape of the imposed disturbance.

Questions concerning the development of nonlinear non-parallel vortices within growing boundary layers were answered by Hall (1988). This numerical investigation showed that as the nonlinear disturbance evolved, the perturbation energy became concentrated in the fundamental and mean flow correction; a conclusion consistent with the weakly nonlinear theory of Hall (1982*b*) valid for small wavelength vortices. It is well known that Görtler vortices set up in an experiment conserve their wavelength as they move downstream. Since the boundary layer itself thickens it follows that the local nondimensional vortex wavenumber becomes large as the vortex develops. Thus the small wavelength limit in the external Görtler problem is appropriate to the ultimate development of any fixed wavelength vortex, and hence sufficiently far downstream in many flows the asymptotic work of Hall (1982*a*, *b*) becomes applicable.

As with all weakly nonlinear investigations, the results of Hall (1982*b*) are valid only within the neighbourhood of the point where the considered perturbation is neutrally stable. For vortices of wavenumber ϵ^{-1}, $0 < \epsilon \ll 1$, their development downstream of the point of neutral stability is dictated by the solution of a pair of coupled nonlinear partial differential equations which adopt a simple asymptotic structure at large values of X, where ϵX ($X = O(1)$) denotes the distance of the vortex downstream of the neutral point. Formally, for large X, Hall & Lakin (1988) showed that this asymptotic structure could be used to deduce the flow configuration for fully nonlinear Görtler vortices, at which point the mean flow correction generated by the presence of the vortices becomes as large as the basic (undisturbed) flow itself. These fully nonlinear vortices, as described by Hall & Lakin (hereafter referred to as HL) are of the type whose stability to travelling wave disturbances will be considered here. For this reason, it is convenient to now briefly describe the main results of HL.

Consider a vortex of wavelength $2\pi\epsilon$ developing in a boundary layer flow at a Görtler number $O(\epsilon^{-4})$, and suppose that the flow is neutrally stable at the downstream position x_n. For $x > x_n$, HL demonstrated that the flow field assumes the structure illustrated in Figure 1, where the vortex activity is restricted to region I and decays exponentially within thin shear layers IIa, b. Suppose that these are located at y_1 and y_2. Then exterior to the shear layers, in zones $IIIa, b$, the vortex is annihilated and the mean flow satisfies the usual boundary layer equations. In the central part of the flow, region I, the mean flow is dictated by a solvability condition on the fundamental component of the vortex and adjusts itself so as to make the fundamental and all harmonics finite within I. The vortex is determined by the mean flow equations, and the positions of the shear layers y_1, y_2 change as they move downstream: in fact they are given by a free boundary value problem derived from the boundary layer equations. Recall that this small wavelength asymptotic approach is relevant to many flows of practical importance for the reasons given previously.

We shall study secondary instabilities of the flow in the shear layers IIa, b. Perturbations which are spanwise periodic travelling waves are imposed on the flow within these zones and their development monitored. These disturbances are $\pi/2$ radians out of phase with the fundamental in the spanwise direction, so that any secondary instabilities which occur produce locally wavy vortex boundaries in IIa, b. At the outset it is not clear that the shear zones should be susceptible to the wavy modes but Hall & Seddougui (1989) (hereafter referred to as HS) presented an argument as to why this should be the case. Basically, they considered a model equation which governs the nonlinear growth of time-dependent Görtler vortices in curved channel flows. The instability of a steady, streamwise independent vortex to travelling vortex-like perturbations was investigated and it was demonstrated that if the travelling perturbations were in phase with the fundamental component of the steady vortex then the flow is always stable whereas if the disturbance were $\pi/2$ out of phase then instability could result. Moreover, if the Görtler number was large, the disturbance was confined to the shear layers and decayed away from the centres of these layers. This analogy with the curved channel flow problem led HS to analyse disturbances in the boundary layer problem which were concentrated in the shear layers IIa, b. As the disturbances decayed exponentially away from the centre of these shear layers, HS deduced that unless exponentially small terms were to be matched the two shear layers could be treated independently.

In HS the wavy perturbations were assumed to have an infinitesimally small amplitude. It was found that in this case these wavy disturbances are fixed by a pair of coupled, second order, linear ordinary differential equations which were solved numerically. Hall & Seddougui found only one neutrally stable travelling wave solution of these equations and this information was used to explain some of the experimental observations of wavy vortices; notably those of Kohama (1987) and Peerhossaini & Wesfreid (1988). However, later studies by Bassom & Seddougui (1989) modified the conclusions of HS by demonstrating that infinitely many neutrally stable modes are possible and properties of these modes in a high–wavenumber limit were described asymptotically.

Here we aim to increase the amplitude of the wavy disturbances to extend the ideas of HS to the nonlinear regime. Importantly, it is found that at the point at which the wavy mode becomes nonlinear, the steady vortex flow as determined by HL is affected by self–interactions of the wavy mode. Additionally, at this stage all the harmonics of the wavy mode become sufficiently large so as to influence the fundamental at leading order. Consequently, we found that this nonlinear problem is governed by a fully coupled, infinite set of differential equations which necessarily have to be solved numerically. Rather than tackle this complete system, we concentrate on a weakly nonlinear limit which enables further analytical progress to be made. The results of this work are considered in the light of the available experimental evidence and conjectures made concerning the results which could be anticipated from a strongly nonlinear calculation.

The procedure adopted in the remainder of the paper is as follows. In section 2 we formulate the problem and develop the defining equations for the nonlinear wavy modes in section 3. We analyse the small amplitude limit of these equations in section 4 and in section 5 we discuss our results, consider the full numerical solution of the problem and draw some brief conclusions.

§2. Formulation of the problem.

The flow considered here is that as described in HL. We study the flow of an incompressible, viscous fluid of density ρ and kinematic viscosity ν over a wall of varying curvature $\frac{1}{R}\chi(X/L)$. Here R is a typical radius of curvature, X denotes distance along the wall and L is a typical lengthscale along the wall. Much of the following detail concerning the basic flow structure in the absence of the imposed travelling disturbances follows very closely that described in HL and HS and so we keep our description of this aspect as brief as possible consistent with the aim of retaining clarity.

The Reynolds number Re for the flow is defined by

$$Re = \frac{U_0 L}{\nu}, \tag{2.1}$$

where U_0 is a typical fluid velocity. The curvature parameter δ is defined by

$$\delta = \frac{L}{R}, \tag{2.2}$$

and our concern is with the case in which $Re \longrightarrow \infty$ whilst the Görtler number G, given by

$$G = 2Re^{\frac{1}{2}}\delta, \tag{2.3}$$

maintains a constant O(1) value. We denote time by T and co-ordinates along the wall, normal to the wall and in the spanwise direction by X, Y and Z respectively. If the corresponding velocity vector is (U, V, W) then we choose nondimensional co-ordinates (x, y, z) and velocities (u, v, w) given by

$$(x,y,z) = L^{-1}\left(X, Re^{\frac{1}{2}}Y, Re^{\frac{1}{2}}Z\right), \qquad (u,v,w) = U_0^{-1}\left(U, Re^{\frac{1}{2}}V, Re^{\frac{1}{2}}W\right). \tag{2.4a,b}$$

In the work considered here we restrict ourselves to examining flows for which $u \longrightarrow 1$ as $y \longrightarrow \infty$, although there is no formal difficulty in extending our treatment to account for more general flows. We write the pressure

$$P = \frac{\rho U_0^2}{Re} p, \tag{2.4c}$$

and substitute the expressions (2.4) into the governing continuity and Navier–Stokes equations. Defining the nondimensional time variable by t, where $t = U_0 L^{-1} T$, we find that when terms of order $Re^{-\frac{1}{2}}$ are neglected in the governing equations, these equations assume the forms

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{2.5a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \tag{2.5b}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{2}G\chi u^2-\frac{\partial p}{\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}, \tag{2.5c}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}. \tag{2.5d}$$

The motivation for the work contained in HL was to obtain a solution of (2.5) which satisfies

$$u=v=w=0 \quad \text{on} \quad y=0, \qquad u\longrightarrow 1 \quad \text{as} \quad y\longrightarrow\infty, \tag{2.6}$$

and which is an asymptotic solution valid in the limit of small vortex wavelength. The Görtler number G expands according to

$$G=G_0\epsilon^{-4}+G_1\epsilon^{-3}+\ldots, \tag{2.7}$$

where ϵ^{-1} $(0<\epsilon\ll 1)$ is the wavenumber of the vortices. In the core part of the boundary layer (region I), the leading order z-independent part of the downstream velocity component $\bar{u}(x,y)$ satisfies

$$G_0\chi\bar{u}\frac{\partial \bar{u}}{\partial y}=1, \tag{2.8}$$

and this flow is driven by the finite-sized steady vortices. These vortices produce a mean flow correction term which is as large as the basic flow which would exist in the absence of the vortices and so the total mean flow field is altered at leading order by the presence of the vortices. The calculation for the vortices within the core region (see Figure 1) reveals that the vortices are trapped between y_1 and y_2 (the values of which are found in the course of the computation) and decay to zero within the thin shear layers IIa, b.

The work given by Davey *et. al.* (1968) showed that Taylor vortex flow is unstable to perturbations differing in phase from the fundamental component of the steady vortex flow by $90°$. In HS, the authors sought a secondary instability that produced locally wavy vortex boundaries in the shear layers IIa, b; akin to the wavy surfaces predicted by Davey *et. al.* in the case of the secondary instability of their Taylor vortex flow. In keeping with this aim, HS examined solutions for out of phase perturbations proportional to

$$E\equiv\exp\left(\frac{i}{\epsilon^2}\int^x K(x)dx-\frac{i\Omega t}{\epsilon^2}\right), \tag{2.9a}$$

where the wavenumber K was expanded as

$$K = K_0 + \epsilon^{\frac{2}{3}} K_1 + \ldots, \tag{2.9b}$$

and Ω is the constant frequency. The wavy vortex mode was assumed to be a short wavelength, high frequency mode which was necessary in order to ensure that this secondary instability was to remain trapped within IIa or IIb, an assumption which is consistent with the available experimental observations. The scales in (2.9) are chosen, from Hall (1982b), so that $\frac{\partial^2}{\partial z^2} \sim \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} = 0$ in the shear layers. The latter scaling ensures that the waves travel with the speed of the fluid in the shear layer, as observed in the work of Kohama (1987).

In HS the secondary instabilities were supposed to be of infinitesimal amplitude so that the steady vortex flow was unaffected by the presence of these wavy modes. Our present aim is to allow this travelling vortex to be of sufficiently large amplitude so that self-interactions of this mode do modify the governing equations for the steady vortex. We then have the case of a fully coupled problem in which the primary vortex and the wavy mode must be determined simultaneously; in contrast to the calculation presented in HS in which the wavy mode was found only after the steady vortex had been fully determined. In the following section we develop the asymptotic expansions for both the vortex and the nonlinear wavy mode under the assumption that the latter is confined to within the shear layer IIa, an analysis which may be trivially modified to account for modes within the other shear layer if so desired.

§3. The asymptotic structure of the wavy modes.

In HL it was demonstrated that the layers IIa, b are of thicknesses $O(\epsilon^{\frac{1}{3}})$ and so in IIa we define the co-ordinate

$$\xi = \frac{(y - y_2)}{\epsilon^{\frac{1}{3}}}, \tag{3.1}$$

where $y_2(x)$ is the location of the zone IIa. Hence, in this layer we replace $\partial/\partial x$ and $\partial/\partial y$ by

$$\frac{\partial}{\partial x} - \frac{y_2'}{\epsilon^{\frac{1}{3}}}\frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{1}{\epsilon^{\frac{1}{3}}}\frac{\partial}{\partial \xi}$$

respectively. The vortex activity is wholly within the core region I between IIa and IIb, and the algebraically decaying vortices in I are reduced to zero exponentially within the shear layers. In the analysis described below we find that the leading order mean flow terms across the whole of the boundary layer are unchanged by the presence of the nonlinear wavy mode terms. Additionally, the steady vortex terms within the core region I are unaffected by the wavy mode, but this latter disturbance does alter the vortex quantities within the shear layers.

We consider the amplitude of the wavy mode to be such that the equations given in HS to determine the steady vortex within the shear layers are modified due to interactions of the wavy disturbances. It is found that when the wavy mode is as large as the steady vortex ($= O(\epsilon^{\frac{4}{3}})$, by HL) this change in the character of the governing equations occurs and we are led to consider total velocity and pressure fields within the shear layer IIa which take the forms

$$
\begin{aligned}
u =& \bar{\bar{u}}_0 + \epsilon^{\frac{2}{3}}\bar{\bar{u}}_1 + \epsilon\bar{\bar{u}}_2 + \epsilon^{\frac{4}{3}}\bar{\bar{u}}_3 + \ldots + \epsilon^{\frac{4}{3}}\left[CU_{0c0} + \epsilon^{\frac{2}{3}}CU_{0c1} + \ldots\right] \\
&+ \epsilon^{\frac{4}{3}}\left[\sum_{n=1}^{\infty}\left(Cu_{0cn} + Su_{0sn}\right)E^n + \epsilon^{\frac{2}{3}}\sum_{n=1}^{\infty}\left(Cu_{1cn} + Su_{1sn}\right)E^n + \ldots + c.c.\right] \\
&+ \epsilon^{\frac{4}{3}}\left[\sum_{n=1}^{\infty}\hat{\hat{U}}_{0n}E^n + \ldots + c.c.\right] + \ldots,
\end{aligned}
\tag{3.2a}
$$

$$
\begin{aligned}
v =& \bar{\bar{v}}_0 + \epsilon^{\frac{2}{3}}\bar{\bar{v}}_1 + \ldots + \epsilon^{-\frac{2}{3}}\left[CV_{0c0} + \epsilon^{\frac{2}{3}}CV_{0c1} + \ldots\right] \\
&+ \epsilon^{-\frac{2}{3}}\left[\sum_{n=1}^{\infty}\left(Cv_{0cn} + Sv_{0sn}\right)E^n + \epsilon^{\frac{2}{3}}\sum_{n=1}^{\infty}\left(Cv_{1cn} + Sv_{1sn}\right)E^n + \ldots + c.c.\right] \\
&+ \left[\sum_{n=1}^{\infty}\hat{\hat{V}}_{0n}E^n + \ldots + c.c.\right] + \ldots,
\end{aligned}
\tag{3.2b}
$$

$$
\begin{aligned}
w =& \epsilon^{-\frac{1}{3}}\left[SW_{0s0} + \epsilon^{\frac{2}{3}}SW_{0s1} + \ldots\right] \\
&+ \epsilon^{-\frac{1}{3}}\left[\sum_{n=1}^{\infty}\left(Cw_{0cn} + Sw_{0sn}\right)E^n + \epsilon^{\frac{2}{3}}\sum_{n=1}^{\infty}\left(Cw_{1cn} + Sw_{1sn}\right)E^n + \ldots + c.c.\right] \\
&+ \epsilon^{-\frac{1}{3}}\left[\sum_{n=1}^{\infty}\hat{\hat{W}}_{0n}E^n + \ldots + c.c.\right] + \ldots,
\end{aligned}
\tag{3.2c}
$$

$$
\begin{aligned}
p =& \epsilon^{-4}\bar{\bar{p}}_0 + \epsilon^{-\frac{10}{3}}\bar{\bar{p}}_1 + \ldots + \epsilon^{-\frac{4}{3}}\left[CP_{0c0} + \epsilon^{\frac{2}{3}}CP_{0c1} + \ldots\right] \\
&+ \epsilon^{-\frac{4}{3}}\left[\sum_{n=1}^{\infty}\left(Cp_{0cn} + Sp_{0sn}\right)E^n + \epsilon^{\frac{2}{3}}\sum_{n=1}^{\infty}\left(Cp_{1cn} + Sp_{1sn}\right)E^n + \ldots + \text{c.c.}\right] \\
&+ \epsilon^{-\frac{4}{3}}\left[\sum_{1}^{\infty}\hat{\hat{P}}_{0n}E^n + \ldots + \text{c.c.}\right] + \ldots.
\end{aligned}
\tag{3.2d}
$$

In (3.2) C and S denote $\cos(z/\epsilon)$ and $\sin(z/\epsilon)$ respectively, all unknown coefficients are functions of x and ξ and *c.c.* denotes complex conjugate. The expansions are more complicated versions of those presented in HS, and we recall that the variation E defined by (2.9) dictates the streamwise development of those parts of the total velocity and pressure fields corresponding to the wavy mode. As is expected, when the amplitude of the wavy mode is decreased from that value implied in (3.2), we retrieve the expansions of HS. We remark that higher harmonics of the spanwise variation lead to terms of smaller sizes than those given in (3.2), see HL. These terms do not enter the determination of the primary vortex or the wavy mode and so have been omitted in (3.2). However, we do observe that all harmonics of the spatial and temporal variation E must be accounted for since they all play nontrivial roles in the subsequent analysis.

We find the unknowns in (3.2) by substituting these forms into (2.5) and equating coefficients of specific terms. The leading order mean flow quantities $\bar{\bar{u}}_0$ and $\bar{\bar{u}}_1$ are found to be precisely as in HL so that

$$
\bar{\bar{u}}_0 = \sqrt{\frac{a + 2y_2}{G_0\chi}}, \qquad \bar{\bar{u}}_1 = \frac{\xi}{\sqrt{G_0\chi(a + 2y_2)}}. \tag{3.3}
$$

Here $a(x)$ is an arbitrary function which may be determined by considering the solutions in region I, see HL. In HL, the fundamental vortex term V_{0c0} was found by deriving a solvability criterion on higher order vortex t s within the shear layer IIa. With the large amplitude wavy disturbance imposed, this solvability condition assumes the form

$$
\begin{aligned}
&3\frac{\partial^2 V_{0c0}}{\partial\xi^2} + \frac{\xi V_{0c0}}{a + 2y_2} + \sqrt{G_0\chi(a + 2y_2)}\frac{\partial\bar{\bar{u}}_3}{\partial\xi}V_{0c0} = \\
&\sum_{n=1}^{\infty}\left(\hat{\hat{W}}_{0n}\left(v_{0sn} - G_0\chi\bar{\bar{u}}_0 u_{0sn}\right)^* + c.c.\right) - G_0\chi\bar{\bar{u}}_0\sum_{n=1}^{\infty}\left(v_{0cn}\frac{\partial\hat{\hat{U}}^*_{0n}}{\partial\xi} + c.c.\right),
\end{aligned}
\tag{3.4a}
$$

where

$$\frac{\partial \bar{\bar{u}}_3}{\partial \xi} = f(x) - \left(\frac{\chi'(a+2y_2)}{3G_0\chi^2} + \frac{b}{\sqrt{G_0\chi(a+2y_2)}} \right) \xi - \frac{V_{0c0}^2}{2\sqrt{G_0\chi(a+2y_2)}}$$
$$+ \frac{1}{2}\int \left(\sum_{n=1}^{\infty} \left(v_{0cn}\frac{\partial u^*_{0cn}}{\partial \xi} + v_{0sn}\frac{\partial u^*_{0sn}}{\partial \xi} + w_{0cn}u^*_{0sn} - w_{0sn}u^*_{0cn} + c.c. \right) \right) d\xi. \tag{3.4b}$$

These equations are derived following procedures identical to those described in HS. Here $b(x)$ is a function arising from the solution in the core, $f(x)$ is a function which can only be found by considering higher order terms (which proves unnecessary here) and an asterisk on a quantity denotes the complex conjugate of that quantity. The equations equivalent to (3.4) for the case of infinitesimally sized wavy vortices, as in HS, may be retrieved by removing the terms comprising products of wavy mode perturbation quantities in $(3.4a, b)$.

Comparing coefficients of $\sin(z/\epsilon)$ in $(2.5a-c)$ and of $\cos(z/\epsilon)$ in $(2.5d)$ yields the following relations between the leading order wavy vortex terms. We obtain

$$\frac{\partial v_{0sn}}{\partial \xi} - w_{0cn} = 0, \tag{3.5a}$$

$$in\left(K_0\bar{\bar{u}}_0 - \Omega\right)u_{0sn} + v_{0sn}\frac{\partial \bar{\bar{u}}_1}{\partial \xi} = -u_{0sn}, \tag{3.5b}$$

$$in\left(K_0\bar{\bar{u}}_0 - \Omega\right)v_{0sn} + G_0\chi\bar{\bar{u}}_0 u_{0sn} = -v_{0sn}, \tag{3.5c}$$

$$in\left(K_0\bar{\bar{u}}_0 - \Omega\right)w_{0cn} = -p_{0sn} - w_{0cn}. \tag{3.5d}$$

Now $(3.5b, c)$ are only consistent if $K_0\bar{\bar{u}}_0 = \Omega$ and then we have

$$w_{0cn} = \frac{\partial v_{0sn}}{\partial \xi}, \qquad u_{0sn} = -\frac{\partial \bar{\bar{u}}_1}{\partial \xi}v_{0sn}, \qquad p_{0sn} = -\frac{\partial v_{0sn}}{\partial \xi}. \tag{3.5e}$$

By an entirely similar procedure we can obtain

$$w_{0sn} = -\frac{\partial v_{0cn}}{\partial \xi}, \qquad u_{0cn} = -\frac{\partial \bar{\bar{u}}_1}{\partial \xi}v_{0cn}, \qquad p_{0cn} = -\frac{\partial v_{0cn}}{\partial \xi}. \tag{3.5f}$$

Substituting into (3.4) and simplifying reveals that the steady fundamental vortex term V_{0c0} satisfies

$$\frac{\partial^2 V_{0c0}}{\partial \xi^2} - \xi V_{0c0}\left[\frac{\chi'(a+2y_2)^{\frac{3}{2}}}{9\chi\sqrt{G_0\chi}} + \frac{b}{3} - \frac{1}{3(a+2y_2)} \right] = \frac{V_{0c0}^3}{6}$$
$$+ \frac{V_{0c0}}{3}\left(\sum_{n=1}^{\infty} \left(|v_{0cn}|^2 + |v_{0sn}|^2\right) \right) - \frac{f\sqrt{G_0\chi(a+2y_2)}}{3}V_{0c0} \tag{3.6}$$
$$+ \frac{1}{3}\sum_{n=1}^{\infty}\left[2\hat{\hat{W}}^*_{0n}v_{0sn} - \sqrt{G_0\chi(a+2y_2)}\frac{\partial \hat{\hat{U}}^*_0}{\partial \xi}v_{0cn} + c.c. \right].$$

Examining terms proportional to E^n in (2.5d) yields the governing equation for the mean flow correction term $\hat{\hat{W}}_{0n}$;

$$\frac{\partial^2\hat{\hat{W}}_{0n}}{\partial\xi^2} - in\left[K_1\bar{\bar{u}}_0 + \frac{\Omega\bar{\bar{u}}_1}{\bar{\bar{u}}_0}\right]\hat{\hat{W}}_{0n} =$$
$$\frac{1}{2}\left(V_{0c0}\frac{\partial^2 v_{0sn}}{\partial\xi^2} - v_{0sn}\frac{\partial^2 V_{0c0}}{\partial\xi^2}\right) + \frac{1}{2}\sum_{k=-\infty}^{\infty}\frac{\partial}{\partial\xi}\left(v_{0ck}\frac{\partial v_{0s(n-k)}}{\partial\xi} - v_{0sk}\frac{\partial v_{0c(n-k)}}{\partial\xi}\right). \tag{3.7}$$

To obtain the equation for the wavy vortex term v_{0sn} we need to consider the governing equations for the higher order terms in the scheme outlined in obtaining (3.5). Following this procedure we find from (2.5b,c) that u_{1sn} and v_{1sn} satisfy

$$\begin{aligned} &in\left(K_0\bar{\bar{u}}_1 + K_1\bar{\bar{u}}_0\right)u_{0sn} + v_{0sn}\frac{\partial\bar{\bar{u}}_3}{\partial\xi} - \hat{\hat{W}}_{0n}U_{0c0} - \frac{\partial^2 u_{0sn}}{\partial\xi^2} \\ &+\sum_{k=-\infty}^{\infty}\left(v_{0sk}\frac{\partial\hat{\hat{U}}_{0(n-k)}}{\partial\xi} - \hat{\hat{W}}_{0k}u_{0c(n-k)}\right) = -u_{1sn} - v_{1sn}\frac{\partial\bar{\bar{u}}_1}{\partial\xi}, \end{aligned} \tag{3.8a}$$

and

$$\begin{aligned} &in\left(K_0\bar{\bar{u}}_1 + K_1\bar{\bar{u}}_0\right)v_{0sn} - \frac{\partial^2 v_{0sn}}{\partial\xi^2} + \frac{\partial p_{0sn}}{\partial\xi} - V_{0c0}\hat{\hat{W}}_{0n} + G_0\chi\bar{\bar{u}}_1 u_{0sn} \\ &-\sum_{k=-\infty}^{\infty}v_{0ck}\hat{\hat{W}}_{0(n-k)} = -G_0\chi\bar{\bar{u}}_0 u_{1sn} - v_{1sn}. \end{aligned} \tag{3.8b}$$

Now using the results (3.5e,f) we find that (3.8a,b) are consistent for the unknowns (u_{1sn}, v_{1sn}) only if

$$\begin{aligned} &\frac{\partial^2 v_{0sn}}{\partial\xi^2} + \xi\left[\frac{1}{3(a+2y_2)} - \frac{\chi'(a+2y_2)^{\frac{3}{2}}}{9\chi\sqrt{G_0\chi}} - \frac{b}{3}\right]v_{0sn} \\ &-\frac{2in}{3}\left[\frac{K_1\sqrt{a+2y_2}}{\sqrt{G_0\chi}} + \frac{\Omega\xi}{a+2y_2}\right]v_{0sn} = \frac{v_{0sn}V_{0c0}^2}{6} \\ &+\frac{v_{0sn}}{3}\sum_{n=1}^{\infty}\left(|v_{0cn}|^2 + |v_{0sn}|^2\right) - \frac{f}{3}\sqrt{G_0\chi(a+2y_2)}v_{0sn} - \frac{2}{3}\hat{\hat{W}}_{0n}V_{0c0} \\ &-\frac{1}{3}\sum_{k=-\infty}^{\infty}\left(\sqrt{G_0\chi(a+2y_2)}v_{0sk}\frac{\partial\hat{\hat{U}}_{0(n-k)}}{\partial\xi} + 2v_{0ck}\hat{\hat{W}}_{0(n-k)}\right). \end{aligned} \tag{3.9a}$$

Using an identical method to determine a solvability condition for the quantities

u_{1cn} and v_{1cn}, we also find that

$$\frac{\partial^2 v_{0cn}}{\partial\xi^2}+\xi\left[\frac{1}{3(a+2y_2)}-\frac{\chi'(a+2y_2)^{\frac{3}{2}}}{9\chi\sqrt{G_0\chi}}-\frac{b}{3}\right]v_{0cn}$$

$$-\frac{2in}{3}\left[\frac{K_1\sqrt{a+2y_2}}{\sqrt{G_0\chi}}+\frac{\Omega\xi}{a+2y_2}\right]v_{0cn}=\frac{v_{0cn}V_{0c0}^2}{6}$$

$$+\frac{v_{0cn}}{3}\sum_{n=1}^{\infty}\left(|v_{0cn}|^2+|v_{0sn}|^2\right)-\frac{f}{3}\sqrt{G_0\chi(a+2y_2)}v_{0cn}-\frac{V_{0c0}}{3}\sqrt{G_0\chi(a+2y_2)}\frac{\partial\hat{\hat{U}}_{0n}}{\partial\xi}$$

$$-\frac{1}{3}\sum_{k=-\infty}^{\infty}\left(\sqrt{G_0\chi(a+2y_2)}v_{0ck}\frac{\partial\hat{\hat{U}}_{0(n-k)}}{\partial\xi}-2v_{0sk}\hat{\hat{W}}_{0(n-k)}\right).$$

(3.9b)

Finally, to close the system, we need to find the equations which determine the harmonic terms $\hat{\hat{U}}_{0n}$. Considering the coefficients of E^n in (2.5a,b), we find that

$$inK_0\hat{\hat{U}}_{0n}+\frac{\partial\hat{\hat{V}}_{0n}}{\partial\xi}=0, \qquad (3.10a)$$

and

$$\frac{\partial^2\hat{\hat{U}}_{0n}}{\partial\xi^2}-in\left(K_1\bar{\bar{u}}_0+\frac{\Omega\bar{\bar{u}}_1}{\bar{\bar{u}}_0}\right)\hat{\hat{U}}_{0n}-\frac{\partial\bar{\bar{u}}_1}{\partial\xi}\hat{\hat{V}}_{0n}=-\frac{1}{\sqrt{G_0\chi\,(a+2y_2)}}\frac{\partial}{\partial\xi}\left(v_{0cn}V_{0c0}\right)$$

$$-\frac{1}{2\sqrt{G_0\chi\,(a+2y_2)}}\frac{\partial}{\partial\xi}\left[\sum_{k=-\infty}^{\infty}\left(v_{0ck}v_{0c(n-k)}+v_{0sk}v_{0s(n-k)}\right)\right].$$

(3.10b)

We now have the complete specification of our problem to determine the nonlinear vortices. The steady vortex term V_{0c0} is given by (3.6), the mean flow $\hat{\hat{W}}_{0n}$ by (3.7), the wavy mode quantities v_{0cn}, v_{0sn} by (3.9) and the harmonic terms $\hat{\hat{U}}_{0n}$ by (3.10). This system comprises an eigenproblem which may be simplified by making the following transformations. We define the functions

$$g_1(x)=\frac{1}{3(a+2y_2)}-\frac{b}{3}-\frac{\chi'(a+2y_2)^{\frac{3}{2}}}{9\chi\sqrt{G_0\chi}},\qquad g_2(x)=\frac{\sqrt{G_0\chi\,(a+2y_2)}}{3}, \qquad (3.11a,b)$$

transform the independent co-ordinate ξ to ξ_1 by defining

$$\xi_1=\xi+\frac{g_2 f}{g_1}, \qquad (3.11c)$$

write

$$K_1^{\dagger}=K_1-\frac{f g_2\Omega\sqrt{G_0\chi}}{g_1\,(a+2y_2)^{\frac{3}{2}}}, \qquad (3.11d)$$

and finally define nondimensional quantities according to

$$\eta =(-g_1)^{\frac{1}{3}}\xi_1, \quad V_{0c0} = \sqrt{6}(-g_1)^{\frac{1}{3}}A, \quad v_{0cn} = (-g_1)^{\frac{1}{3}}B_{0cn}, \quad v_{0sn} = (-g_1)^{\frac{1}{3}}B_{0sn},$$

$$\hat{\hat{W}}_{0n} = (-g_1)^{\frac{2}{3}}C_{0n}, \quad \frac{\partial\hat{\hat{U}}_{0n}}{\partial\eta} = \frac{(-g_1)^{\frac{1}{3}}}{g_2}D_{0n}, \quad \hat{\Omega} = \frac{\Omega}{(-g_1)(a+2y_2)},$$

and
$$\hat{K}_1 = \frac{\sqrt{a+2y_2}}{\sqrt{G_0\chi}}(-g_1)^{-\frac{2}{3}}K_1^\dagger. \tag{3.12}$$

Inspection of the resulting equations reveals that necessarily we must have

$$B_{0c(2j-1)} = B_{0s(2j)} = C_{0(2j)} = D_{0(2j-1)} = 0,$$

for all integers j, and then if we define

$$B_{0c(2j)} = B_{2j}, \qquad B_{0s(2j-1)} = B_{2j-1}, \qquad C_{0(2j-1)} = C_j, \qquad D_{0(2j)} = D_j, \tag{3.13a}$$

note that by the conjugacy relationships implied by (3.2) we have

$$B_{-j} = B_j^*, \qquad D_{-j} = D_j^*, \qquad C_j = C_{1-j}^*, \tag{3.13b}$$

and define the quantities $B_0 = D_0 = 0$ for completeness, we obtain the system

$$\begin{aligned}\frac{d^2A}{d\eta^2} - \eta A =& A^3 + \frac{A}{3}\left(\sum_{r=1}^{\infty}|B_r|^2\right) \\ &+ \frac{1}{3\sqrt{6}}\sum_{j=1}^{\infty}\left[2B_{2j-1}C_{1-j} - 3B_{2j}D_{-j} + c.c.\right],\end{aligned} \tag{3.14a}$$

$$\begin{aligned}\frac{d^2B_{2j-1}}{d\eta^2} - \eta B_{2j-1} - \frac{2i}{3}(2j-1)(\hat{\Omega}\eta + \hat{K}_1)B_{2j-1} = B_{2j-1}A^2 \\ + \frac{B_{2j-1}}{3}\left(\sum_{r=1}^{\infty}|B_r|^2\right) - \frac{2\sqrt{6}}{3}C_jA - \frac{1}{3}\sum_{r=-\infty}^{\infty}\left(3B_{2r-1}D_{j-r} + 2B_{2r}C_{j-r}\right),\end{aligned} \tag{3.14b}$$

$$\begin{aligned}\frac{d^2B_{2j}}{d\eta^2} - \eta B_{2j} - \frac{2i}{3}(2j)(\hat{\Omega}\eta + \hat{K}_1)B_{2j} = B_{2j}A^2 \\ + \frac{B_{2j}}{3}\left(\sum_{r=1}^{\infty}|B_r|^2\right) - \sqrt{6}D_jA - \frac{1}{3}\sum_{r=-\infty}^{\infty}\left(3B_{2r}D_{j-r} - 2B_{2r-1}C_{j-r+1}\right),\end{aligned} \tag{3.14c}$$

$$\frac{d^2C_j}{d\eta^2} - i(2j-1)(\hat{\Omega}\eta + \hat{K}_1)C_j = \frac{\sqrt{6}}{2}\left(A\frac{d^2B_{2j-1}}{d\eta^2} - B_{2j-1}\frac{d^2A}{d\eta^2}\right)$$
$$+\frac{1}{2}\frac{d}{d\eta}\left(\sum_{r=-\infty}^{\infty}\left(B_{2r}\frac{dB_{2j-1-2r}}{d\eta} - B_{2r-1}\frac{dB_{2j-2r}}{d\eta}\right)\right), \tag{3.14d}$$

and

$$\frac{d^2D_j}{d\eta^2} - 2ij(\hat{\Omega}\eta + \hat{K}_1)D_j =$$
$$-\frac{1}{3}\frac{d^2}{d\eta^2}\left[\frac{1}{2}\sum_{r=-\infty}^{\infty}(B_{2r}B_{2j-2r} + B_{2r-1}B_{2j-2r+1}) + \sqrt{6}B_{2j}A\right]. \tag{3.14e}$$

As yet we have not addressed the problem of specifying the boundary conditions pertinent to the system (3.14). Firstly, we anticipate that the wavy vortex be constrained to lie completely within the shear layer IIa and this requires that B_j, C_j and D_j decay exponentially as $|\eta| \longrightarrow \infty$. Secondly, the steady vortex term A must match with the core region solution for the strongly nonlinear vortex within I. This, as in HL, implies that $A \sim (-\eta)^{\frac{1}{2}}$ as $\eta \longrightarrow -\infty$. Finally, all the vortex activity is confined within the core I and the shear layer IIa so that in order that in $IIIa$ the mean flow is governed by the usual boundary layer equations we impose the requirement $A \longrightarrow 0$ as $\eta \longrightarrow \infty$; indeed by (3.14$a$) we have $A \propto Ai(\eta)$ in this limit where Ai is the Airy function, see Abramowitz & Stegun (1964).

To conclude this section, we note that the nonlinear wavy modes within the shear layer IIa are determined by the functions B_j, C_j, D_j which are defined in (3.12) and (3.13). These terms, together with the steady vortex quantity A, satisfy equations (3.14) with associated boundary conditions

$$A \propto Ai(\eta) \qquad \text{as} \qquad \eta \longrightarrow \infty, \tag{3.15a}$$

$$A \sim (-\eta)^{\frac{1}{2}} \qquad \text{as} \qquad \eta \longrightarrow -\infty, \tag{3.15b}$$

and

$$B_j, C_j, D_j \longrightarrow 0 \qquad \text{as} \qquad |\eta| \longrightarrow \infty. \tag{3.15c}$$

In general, the solution of this system is highly complicated numerical task and in the following section we address this problem in a weakly nonlinear limit which permits additional analytical progress.

§4. Weakly nonlinear theory.

We recall that in HS the infinitesimally small wavy mode problem was governed by the system of equations (3.14) for the functions A, B_1 and C_1 when all the nonlinear terms (except those involving the function A) are omitted. HS used a numerical scheme to search for eigenvalues of this linearised problem but located only one such pair. However, HS suggested that further eigenvalue pairs with $\hat{K}_1$ and $\hat{\Omega}$ real (and so corresponding to neutrally stable wavy modes) may be possible and Bassom & Seddougui (1989) found eight such pairs in all. Further, these latter authors also demonstrated that, plausibly, infinitely many pairs exist and they derived the asymptotic structure of the neutrally stable modes in a high-wavenumber, high-frequency limit. The eight eigenpairs identified by Bassom & Seddougui (hereafter BS) are summarised in Table 1.

Our analysis of the full equations (3.14) is concerned with the problem in which the wavy mode is of amplitude $O(h)$, $h \ll 1$, relative to the scalings implied in (3.2). In this case, in keeping with the usual weakly nonlinear approach, we anticipate from (3.14) that the perturbation quantities are changed by $O(h^2)$ from their linear values, and we write

$$\begin{aligned} A &= A_0 + h^2 A_1 + \dots, \quad B_1 = hB_{10} + h^3 B_{11} + \dots, \quad B_2 = h^2 B_{20} + \dots, \\ C_1 &= hC_{10} + h^3 C_{11} + \dots, \quad C_2 = h^2 C_{20} + \dots, \quad D_1 = h^2 D_{10} + \dots, \\ \hat{\Omega} &= (\hat{\Omega})_L + h^2 \Omega_p + \dots, \quad \hat{K}_1 = (\hat{K}_1)_L + h^2 K_p. \end{aligned} \tag{4.1}$$

The remaining terms in (3.14) are sufficiently small so as to become negligible in the ensuing analysis.

Substituting (4.1) in (3.14) and (3.15) gives, at leading orders in h, the linear eigenvalue problem described by HS. In this paper, together with BS, figures of the linear eigenfunctions B_{10} and C_{10} appropriate to some of the eigenpairs in Table 1 are shown. We notice from (3.14a), (3.15a,b) that the steady vortex profile A_0 satisfies

$$\frac{d^2 A_0}{d\eta^2} - \eta A_0 = A_0^3,$$

$$A_0 \longrightarrow \lambda Ai(\eta) \quad \text{as} \quad \eta \longrightarrow \infty, \qquad A_0 \sim (-\eta)^{\frac{1}{2}} \quad \text{as} \quad \eta \longrightarrow -\infty.$$

Further, the results of Hastings & McLeod (1980) demonstrate that $\lambda = \sqrt{2}$ and then the function A_0 may be evaluated by using the asymptotic condition as $\eta \longrightarrow \infty$ and integrating in the direction of decreasing η using a Runge–Kutta method, see HS.

At next order in h when (4.1) is inserted in (3.14), we obtain

$$\left(\frac{d^2}{d\eta^2}-\eta\right)A_1-3A_0^2A_1=\frac{1}{3}A_0|B_{10}|^2+\frac{2}{3\sqrt{6}}\left(B_{10}C_{10}^*+c.c.\right), \tag{4.2a}$$

$$\frac{d^2B_{11}}{d\eta^2}+a_1(\eta)B_{11}+a_2(\eta)C_{11}=\Omega_p\mathcal{L}_1(\eta)+K_p\mathcal{L}_2(\eta)+\mathcal{L}_3(\eta), \tag{4.2b}$$

$$\begin{aligned}\frac{d^2C_{11}}{d\eta^2}+a_3(\eta)B_{11}+a_4(\eta)C_{11}&=\Omega_p\left[i\eta C_{10}+\frac{\sqrt{6}A_0\mathcal{L}_1(\eta)}{2}\right]\\&+K_p\left[iC_{10}+\frac{\sqrt{6}A_0\mathcal{L}_2(\eta)}{2}\right]+\frac{\sqrt{6}A_0}{2}\mathcal{L}_3(\eta)+\mathcal{L}_4(\eta),\end{aligned} \tag{4.2c}$$

where

$$a_1=-\eta\left(1+\frac{2i\hat{\Omega}_L}{3}\right)-\frac{2i(\hat{K}_1)_L}{3}-A_0^2, \qquad a_2=\frac{2\sqrt{6}}{3}A_0, \tag{4.3a, b}$$

$$a_3=-\frac{i\sqrt{6}}{3}\left(\hat{\Omega}_L\eta+(\hat{K}_1)_L\right)A_0, \qquad a_4=-i\left(\hat{\Omega}_L\eta+(\hat{K}_1)_L\right)+2A_0^2. \tag{4.3c, d}$$

Here, we find that $\mathcal{L}_1,\ldots,\mathcal{L}_4$ are defined by

$$\mathcal{L}_1\equiv\frac{2i}{3}\eta B_{10}, \qquad \mathcal{L}_2\equiv\frac{2i}{3}B_{10}, \tag{4.4a, b}$$

$$\mathcal{L}_3\equiv 2A_0A_1B_{10}+\frac{B_{10}|B_{10}|^2}{3}-\frac{2\sqrt{6}}{3}C_{10}A_1-B_{10}^*D_{10}-\frac{2}{3}B_{20}C_{10}^*, \tag{4.4c}$$

$$\begin{aligned}\mathcal{L}_4\equiv&-\sqrt{6}A_1A_0^2B_{10}-2A_0A_1C_{10}+\frac{i\sqrt{6}}{3}\left(\hat{\Omega}_L\eta+\left(\hat{K}_1\right)_L\right)A_1B_{10}-\frac{B_{10}A_0|B_{10}|^2}{\sqrt{6}}\\&-\frac{B_{10}}{3}\left(B_{10}C_{10}^*+2B_{10}^*C_{10}\right)-i\left(\hat{\Omega}_L\eta+\left(\hat{K}_1\right)_L\right)B_{10}^*B_{20}\\&-\frac{\sqrt{6}}{3}A_0B_{20}C_{10}^*+\frac{D_{10}A_0B_{10}^*\sqrt{6}}{2},\end{aligned} \tag{4.4d}$$

where, from HS, we know that the functions B_{10} and C_{10} satisfy the homogeneous forms of equations (4.2b, c). Additionally, from (3.14) we obtain

$$\frac{d^2B_{20}}{d\eta^2}-\eta B_{20}-\frac{4i}{3}\left(\hat{\Omega}_L\eta+\left(\hat{K}_1\right)_L\right)B_{20}-A_0^2B_{20}=\frac{2}{3}B_{10}C_{10}-\sqrt{6}D_{10}A_0, \tag{4.5a}$$

and

$$\frac{d^2D_{10}}{d\eta^2}-2i\left(\hat{\Omega}_L\eta+\left(\hat{K}_1\right)_L\right)D_{10}=-\frac{1}{3}\frac{d^2}{d\eta^2}\left[\frac{1}{2}\left(B_{10}\right)^2+\sqrt{6}B_{20}A_0\right]. \tag{4.5b}$$

To ensure the disturbance is confined within the shear layer we demand A_1, B_{11}, C_{11}, B_{20} and $D_{10} \longrightarrow 0$ as $|\eta| \longrightarrow \infty$.

The homogeneous forms of (4.2b,c) are solved in HS. Then, as is the normal procedure within a weakly nonlinear analysis, (4.2b,c) only have a suitable solution if a certain compatibility condition is satisfied. This condition leads to the specification of the correction terms K_p and Ω_p within the wavenumber and frequency expansions detailed in (4.1). To derive our compatibility requirement we first need to consider the system adjoint to the homogeneous forms of (4.2b,c). The adjoint functions $(G_1(\eta), G_2(\eta))$ satisfy the coupled equations

$$\frac{d^2G_1}{d\eta^2} + a_1(\eta)G_1 + a_3(\eta)G_2 = 0, \tag{4.6a}$$

$$\frac{d^2G_2}{d\eta^2} + a_2(\eta)G_1 + a_4(\eta)G_2 = 0, \tag{4.6b}$$

with associated boundary conditions of G_1, $G_2 \longrightarrow 0$ as $|\eta| \longrightarrow \infty$. Multiplying (4.2$b$) by $G_1(\eta)$, (4.2c) by $G_2(\eta)$, adding the resulting equations and integrating by parts, reveals that for a satisfactory solution of (4.2b,c) to exist, we require

$$\int_{-\infty}^{\infty} (G_1N_1 + G_2N_2)\, d\eta = 0, \tag{4.7}$$

where N_1 and N_2 are the right hand sides of (4.2b,c) respectively.

On substituting (4.4) into (4.7) we eventually obtain a relationship between the wavenumber correction K_p and the frequency correction Ω_p of the type

$$1 + z_1\Omega_p + z_2K_p = 0, \tag{4.8}$$

where z_1, z_2 are complex numbers expressible as quotients of integrals. In particular, $z_1 = I_1/I_3$ and $z_2 = I_2/I_3$ where

$$I_1 = \int_{-\infty}^{\infty} \left[\mathcal{L}_1G_1 + \left(i\eta C_{10} + \frac{\sqrt{6}A_0\mathcal{L}_1}{2}\right) G_2\right] d\eta, \tag{4.9a}$$

$$I_2 = \int_{-\infty}^{\infty} \left[\mathcal{L}_2G_1 + \left(iC_{10} + \frac{\sqrt{6}A_0\mathcal{L}_2}{2}\right) G_2\right] d\eta, \tag{4.9b}$$

and

$$I_3 = \int_{-\infty}^{\infty} \left[\mathcal{L}_3G_1 + \left(\frac{\sqrt{6}A_0\mathcal{L}_3}{2} + \mathcal{L}_4\right) G_2\right] d\eta. \tag{4.9c}$$

Before presenting results of the computations, we observe that we may easily generalise our analysis to allow the wavy perturbation to evolve on a slow lengthscale so that the growth or decay of a non–neutral disturbance may be monitored. Formally, if we consider the neighbourhood of a point $x = x_n$ (at which an infinitesimal wavy vortex is neutrally stable) and allow a constant frequency disturbance to develop on the lengthscale X, where $X = \epsilon^{-\frac{4}{5}}h^{-2}(x - x_n)$, $h \ll 1$, then the result of repeating the previous analysis for a disturbance of amplitude $h\hat{A}(X)$ relative to the scalings in (3.2) is the evolution equation

$$\frac{d\hat{A}}{dX} = -\frac{iz_1}{z_2}\Omega_p \hat{A} - \frac{i}{z_2}|\hat{A}|^2\hat{A}. \tag{4.10}$$

We may then easily obtain the equation for the development of the wavy disturbance amplitude

$$\frac{d}{dX}\left(|\hat{A}|^2\right) = 2\mathrm{Re}\left(\frac{-iz_1}{z_2}\right)\Omega_p|\hat{A}|^2 + 2\mathrm{Re}\left(\frac{-i}{z_2}\right)|\hat{A}|^4. \tag{4.11}$$

To effect the necessary calculations we used the method described by HS to obtain the linear eigenfunctions B_{10} and C_{10}. To fix a normalisation for the problem, we elected to scale these eigenfunctions such that $\max\{|B_{10}| : |\eta| < \infty\} = 1$. The technique of HS is easily adapted to solve the adjoint system (4.6) and essentially consists of the following. Equations (4.6) were written as a set of four first order differential equations, which was solved by using a fourth order Runge–Kutta scheme. Since (4.6) is the system adjoint to the problem specified in HS, the two problems have the same eigenvalues and the integration procedures for (4.6) were started at $\eta = -\infty$ and at $\eta = \infty$ and both continued to $\eta = 0$. The detailed asymptotic conditions on G_1 and G_2 for large $|\eta|$ follow in a similar manner to those given in HS and so yield the initial conditions for the integrations. The two integration procedures from $\pm\infty$ were matched across $\eta = 0$, and hence the adjoint functions G_1, G_2 determined.

In order to obtain the integrands in (4.9) we needed to find the functions A_1, B_{20} and D_{10} in addition to those already computed. The steady vortex quantity A_1, which satisfies $|A_1| \longrightarrow 0$ as $|\eta| \longrightarrow \infty$ was evaluated using a finite differencing procedure on (4.2a). We observe from (4.5) that the remaining wavy vortex terms D_{10} and B_{20} are governed by coupled equations and these functions were found also by employing finite difference methods. Once these flow quantities were all determined, the necessary integrations to evaluate (4.9) were effected by use of trapezoidal integrations and the results so obtained were checked to ensure that they were independent of the step lengths used within the Runge–Kutta and finite differencing procedures and also independent of the (large) values of $|\eta|$ at which the various asymptotic behaviours as $|\eta| \longrightarrow \infty$ were assumed to be valid.

The results of these computations and their implications are considered in the following section.

§5. Results and discussion.

The numerical scheme outlined above was used to determine the coefficients in the amplitude equation (4.11). This equation demonstrates that the weak nonlinearity of the problem allows the existence of a threshold equilibrium amplitude $\hat{A}_e$ given by

$$|\hat{A}_e|^2 = -\left(\frac{\mathrm{Re}\,(-iz_1 z_2^{-1})}{\mathrm{Re}\,(-iz_2^{-1})}\right)\Omega_p. \tag{5.1}$$

The relevant values of the quantities contained within (5.1) are summarised in Table 2, where we list those corresponding to the first five linearly neutrally stable modes given in Table 1. Eigenfunctions for the problem (4.2)–(4.5) in the cases $\hat{\Omega}_L = 0.372$ and $\hat{\Omega}_L = 0.659$ are shown in Figures 2 and 3 respectively.

Inspection of the complex–valued eigenfunctions reveals that the imaginary parts are pictorially similar to the real parts and so we elect to concentrate upon the latter in presenting the eigenfunctions of the problem. Observation of the eigensolutions for the first five linearly neutrally stable modes shows that the eigenfunctions G_1, G_2, B_{20}, D_{10} have the same trends as the linear eigensolutions B_{10}, C_{10} described by Bassom & Seddougui (1989). Thus, the eigenfunctions are oscillatory in nature and centred about a negative η position. Moreover, this position is close to $\eta = -(\hat{K}_1)_L/\hat{\Omega}_L$. We also note that the number of oscillations of the complex eigenfunctions increases for each eigenmode with the eigenfunctions corresponding to the choice $((\hat{K}_1)_L, \hat{\Omega}_L) = (0.690, 0.372)$ having the least number of oscillations.

Since the coefficient of the nonlinear term in (4.11) is negative for each of the cases considered in Table 2, interest is concentrated on supercritical disturbances when $\Omega_p < 0$. Then, on the basis of linear theory, the wavy modes of Table 2 are unstable, but the nonlinearity in (4.11) ensures that the equilibrium amplitude $\hat{A}_e$ is stable in the sense that for $\hat{A} < \hat{A}_e$ the disturbance will tend to grow whereas for $\hat{A} > \hat{A}_e$ it will decay until $\hat{A} = \hat{A}_c$. Of course, here we have only discussed the results for the five modes of lowest frequencies and from the work of Bassom & Seddougui (1989) we know that it is probable that there are infinitely many more modes. However, it is the case that in the development of the flow over a curved wall the lower frequency vortices are likely to be the more important: an aspect which we consider in more detail later.

The procedure described above sought a description of the evolution of a wavy mode in terms of the perturbation Ω_p of the non-dimensional frequency away from the linear neutral values $\left(\hat{\Omega}\right)_L$. We remark that in order to translate our findings for the case of any particular curved wall of prescribed curvature $\chi(x)$, we may only determine the dimensional wavenumber K_1 and frequency Ω once the scaling quantities in (3.11) and (3.12) have been calculated. To do this requires an application of the computation described by HL for the particular $\chi(x)$. In addition to their account of this numerical calculation, HL investigated two asymptotic limits. Firstly, solutions were determined for their free–boundary problem for x close to the linear neutral position $x = x_*$, and secondly, for x far downstream of this point. In the latter case it was assumed that the curvature χ increases at least as quickly as $x^{\frac{1}{2}}$ for $x \gg 1$.

Rather than attempt to pursue the HL calculation for any specific boundary layer flows here, it was felt that it would be more instructive to draw general conclusions concerning nonlinear wavy vortices which would hold for large classes of flows. This is largely motivated by the fact that all the experimental papers relating to wavy vortices within boundary layer flows over curved walls contain insufficient detail to enable the dimensionless values of $\hat{K}_1$ and $\hat{\Omega}$ to be computed. Hence any results we may draw concerning specific flows could not be easily related to practical observations.

We now turn to consider the behaviour of the frequency of the neutrally stable nonlinear modes in each of the asymptotic limits considered by HL. As $x \longrightarrow x_*$, where we recall the definition that x_* is the location where the steady Görtler vortex is linearly neutrally stable, we know that the two shear layers merge. Then, following the notation of HS, if we denote by Ω_T and Ω_B the (dimensional) frequencies of the neutral modes in the upper and lower shear layers respectively, we have from (3.12) that as $x \longrightarrow x_*, \Omega_T \longrightarrow \Omega_B$. We now suppose that for large $x, \chi \sim x^M$, $M > \frac{1}{2}$. This choice of curvature is again chosen to illustrate the type of behaviour we may observe in a practical situation rather than as representing any one particular boundary layer flow. In this case, the behaviour of the functions a, y_1, y_2, b are as given by HL and by routine application of these forms, we may easily show, as in HS, that for $x \gg 1$,

$$\Omega_T \sim \frac{\hat{\Omega}}{9} M G_0^2 x^{2M-1}, \qquad \Omega_B \sim \frac{1}{3}\hat{\Omega}. \tag{5.2}$$

Consequently, as x increases, the frequency of the neutral wavy mode confined to the upper shear layer increases whereas that of the mode in the lower layer tends to a constant value. Thus the modes can be categorised as being of high and low frequency respectively.

We remark that the instability process detailed in section 3 for the upper layer is also perfectly possible within the lower layer IIb, and if exponentially small terms may be neglected, the two shear layers are completely independent for the unstable travelling waves decay exponentially away from the centres of IIa, b. Theoretically at least, our analysis could be adapted to prove that an O(1) disturbance in one shear layer provokes an exponentially small response in the other.

In a practical situation, it is often found that a wavy mode of a predetermined frequency is imposed upon the flow and the development of the disturbance monitored as it moves downstream. Then we anticipate that the two shear layers will breakdown at different x–locations. As discussed by HS, since the downstream velocity component of the the basic state is larger within the upper layer IIa, it is expected that this layer is the first to become unstable. Further, as the wavenumber $K(x)$ relating to a fixed frequency perturbation will be different in the two shear layers, the modes in the layers move with different wavespeeds. Close to the linear neutral position, the shear layers meet and so if breakdown occurs close to this point then the structures within the two layers will be very similar. Far downstream, the layer IIa moves into the freestream and the downstream velocity component tends to the freestream speed. The lower layer sinks towards the bounding wall so the fluid velocity there approaches zero. Thence if the stationary vortices develop over a sufficiently lengthy interval before breakdown occurs, the mode in the upper layer has speed close to that of the freestream, whilst the one in IIb has a very small speed.

We now consider the implications of our analysis for an experimental setting. We note from (5.2) that for the case of a mode of prescribed dimensional frequency Ω imposed upon the flow upstream of the linear neutral point, as we move far downstream (for the case of $\chi \sim x^M$) the non-dimensional frequency $\hat{\Omega}$ for vortices confined to the upper layer IIa tends towards zero. Our results obtained here demonstrate, as previously mentioned, the existence of a stable, threshold equilibrium amplitude, at least for the weakly nonlinear modes close (in parameter space) to one of the first five linearly neutral disturbances. As the perturbation propagates downstream, the non-dimensional frequency $\hat{\Omega}$ of modes trapped within the upper shear layer decreases and our analysis indicates that there are no unstable non-zero equilibrium amplitudes within this frequency regime. Hence, we would anticipate that the low-frequency (low $\hat{\Omega}$) modes investigated in this paper are likely to be the most important in a practical experiment and that finite amplitude wavy vortices are likely to be observable in practice.

In this paper we have chosen not to pursue an extensive solution of (3.14) away from the domain where the weakly nonlinear theory is applicable. In such a calculation, all the unknowns in (3.14) are potentially important and would necessarily have to be accounted for in a fully rational way. Preliminary studies of the computation of the full system (3.14) and (3.15) suggest that this task is far from easy. It would be interesting to make a careful examination of these equations to determine the characteristics of strongly nonlinear wavy modes. However, here we have elected to deal only with the weakly nonlinear approach. The results of this analysis suggest that for numerical computations based on the full version of (3.14) it can be speculated that for each non-dimensional low frequency $\hat{\Omega}$ there is a multitude of possible vortex states, each stable and with a different wavelength. Equivalently, for a wavy vortex of prescribed streamwise wavenumber, there could conceivably be a family of possible non-dimensional frequencies for this disturbance. A thorough description of the possible solutions of (3.14) is eagerly awaited.

Papers by, among others, Bippes & Görtler (1972), Wortmann (1969) and Bippes (1978) describe some experimental investigations of the secondary instability of Görtler vortices. Recently, work by Kohama (1987) and Peerhossaini & Wesfreid (1988) have given more details concerning this secondary breakdown than did the aforementioned authors. Kohama (1987) performed experiments on a NASA laminar flow wing. He considered the breakdown only in the upper parts of the vortices and gave measurements of the speeds of the wavy modes at various downstream locations. Peerhossaini & Wesfreid (1988) examined a boundary layer over a concave section of a curved channel. They reported that when the secondary instability first manifested itself, it did so at the top of the steady Görtler vortices and that further downstream another secondary mode was observed near the wall. Both instabilities had wavy boundaries and so, like HS, we conclude that these modes are formed as a result of the mechanism considered in this paper. This experimental work of Peerhossaini & Wesfreid (1988) lends credence to our conclusion that according to weakly nonlinear theory the finite amplitude wavy vortices are stable and hence expected to be obtainable in practice. Interestingly, Peerhossaini & Wesfreid also commented that as the disturbance developed downstream no coherent frequency of the mode was apparent. Possibly, in this case, the weakly nonlinear approach is no longer tenable and the full system (3.14) is appropriate; we then have the possibility of the existence of a whole family of modes with different frequencies as discussed above. This may explain the experimental observation of Peerhossaini & Wesfreid (1988).

To conclude, we observe that we have shown that according to our weakly nonlinear theory, for non-dimensional frequencies close to the linear neutral values, finite amplitude, stable equilibrium wavy modes exist. Our results are consistent with the available experimental observations; however, the lack of detail given in the accounts of this practical work means that precise comparisons between those studies and our findings is rendered near impossible. Of interest would be the consideration of the strongly nonlinear system (3.14) although we ourselves are not planning to pursue this aspect further. However, we do feel that this investigation merits attention at some stage. We finally note that other instability processes could give rise to three-dimensionality and time-dependence in the latter stages of vortex development; these would most likely be Rayleigh instabilities associated with the spanwise locally inflexional velocity profiles, or Tollmien- Schlichting waves, both of which need to be thoroughly studied to enable a more complete picture of the transition process to be constructed.

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$\hat{K}_1$	0.690	2.900	4.156	5.435	7.53	9.60	11.4	15.7
$\hat{\Omega}$	0.372	0.659	0.742	0.795	1.00	1.17	1.27	1.60

Table 1. Values of $\hat{K}_1$ and $\hat{\Omega}$ for which infinitesimally small wavy vortex modes are neutrally stable, from Bassom & Seddougui (1989).

$\hat{K}_L$	$\hat{\Omega}_L$	$-2iz_1/z_2$	$-2i/z_2$	$\hat{A}_e/(-\Omega_p)^{1/2}$
0.6895	0.3715	$(-2.82, 2.01)$	$(-0.187, 0.345)$	3.89
2.900	0.6587	$(-4.21, 5.08)$	$(-1.019, -6.089 \times 10^{-2})$	2.033
4.156	0.7416	$(-5.64, 6.70)$	$(-1.044, -1.720)$	2.325
5.435	0.7950	$(-7.06, 10.09)$	$(-0.121, -3.997)$	7.652
7.527	0.9976	$(-4.57, 11.86)$	$(-1.335, -4.461)$	1.849

Table 2. Quantities defined in (4.11) and (5.1) appropriate to the weakly nonlinear analysis of section 4.

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FIGURE CAPTIONS

Figure 1. The different regions beyond the downstream position of neutral stability.

Figure 2. Eigenfunctions for the weakly nonlinear problem of §4 as defined by (4.1)–(4.4) when $(\hat{K}_1)_L = 0.690$, $\hat{\Omega}_L = 0.372$. (*a*) $Re(B_{10})$; (*b*) $Re(C_{10})$; (*c*) $Re(G_1)$; (*d*) $Re(G_2)$; (*e*) A_1; (*f*) $Re(B_{20})$ and (*g*) $Re(D_{10})$.

Figure 3. Eigenfunctions for the weakly nonlinear problem of §4 as defined by (4.1)–(4.4) when $(\hat{K}_1)_L = 2.900$, $\hat{\Omega}_L = 0.659$. (*a*) $Re(B_{10})$; (*b*) $Re(C_{10})$; (*c*) $Re(G_1)$; (*d*) $Re(G_2)$; (*e*) A_1; (*f*) $Re(B_{20})$ and (*g*) $Re(D_{10})$.

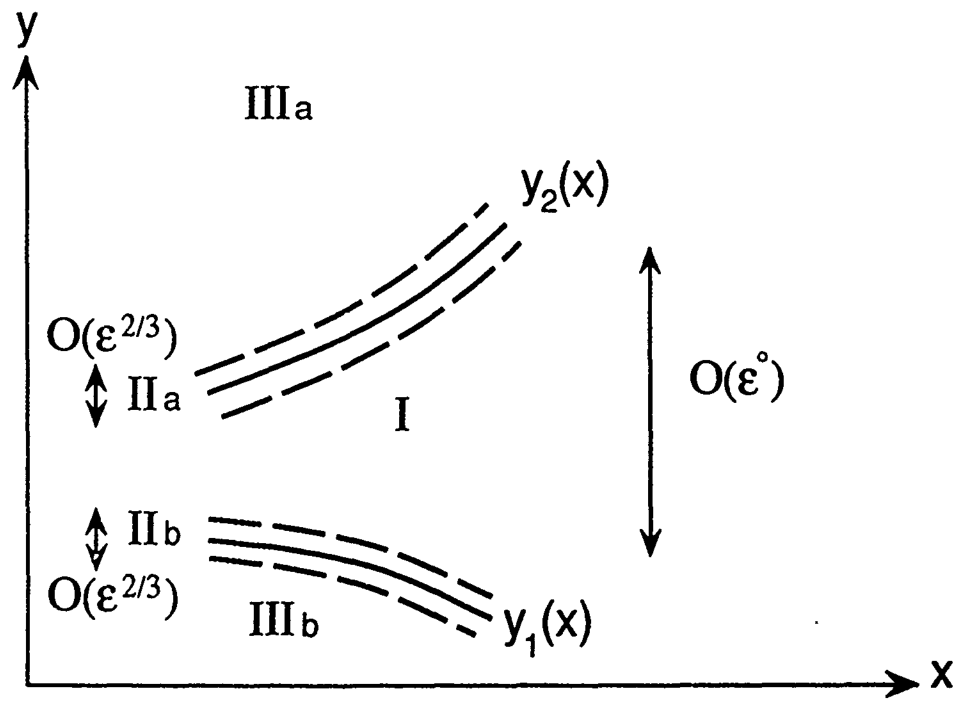

y
IIIa
$y_2(x)$
$O(\varepsilon^{2/3})$
IIa
I
$O(\varepsilon^{\circ})$
IIb
$O(\varepsilon^{2/3})$
IIIb
$y_1(x)$
x

Figure 1

Figure 2a

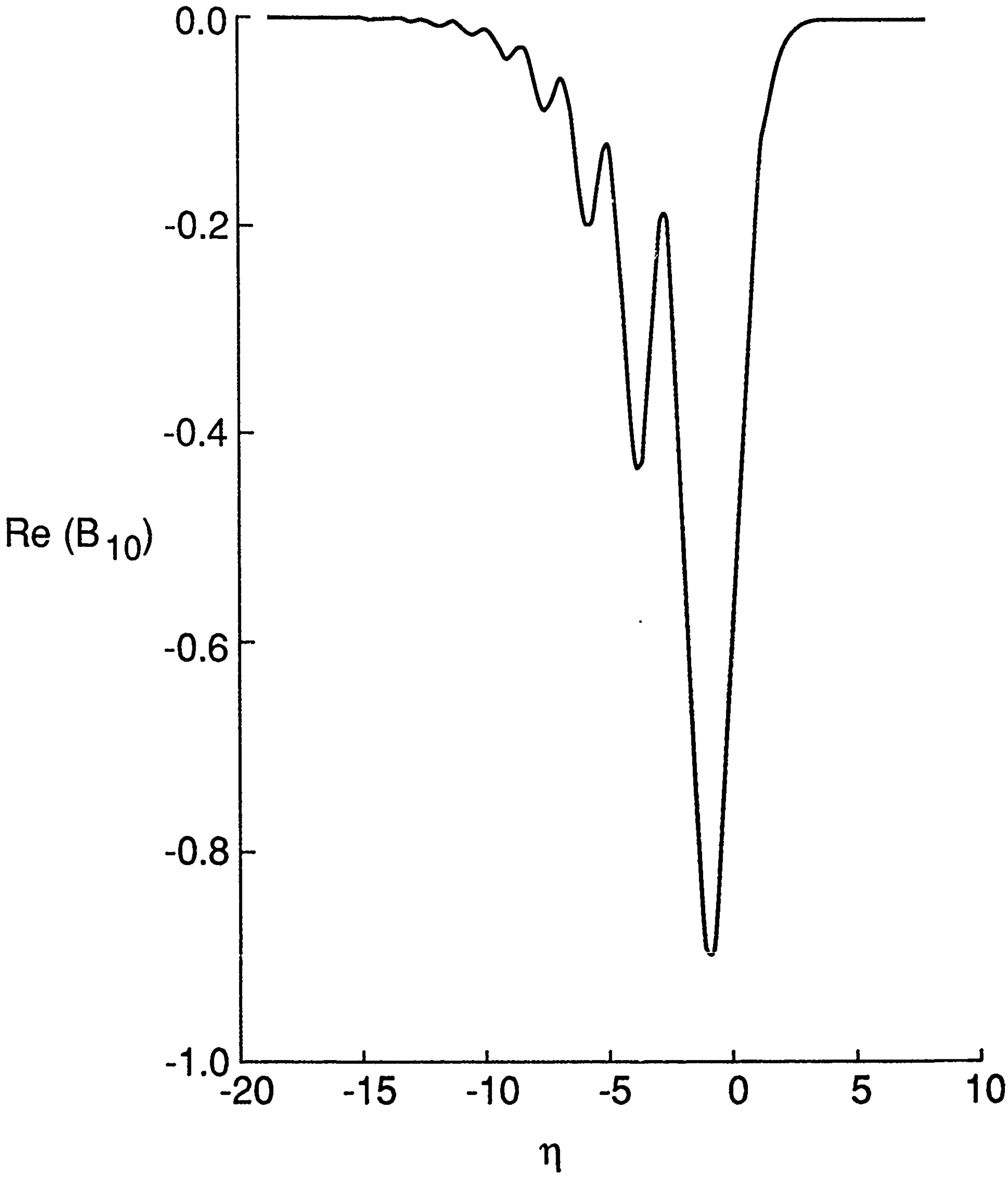


Figure 2b

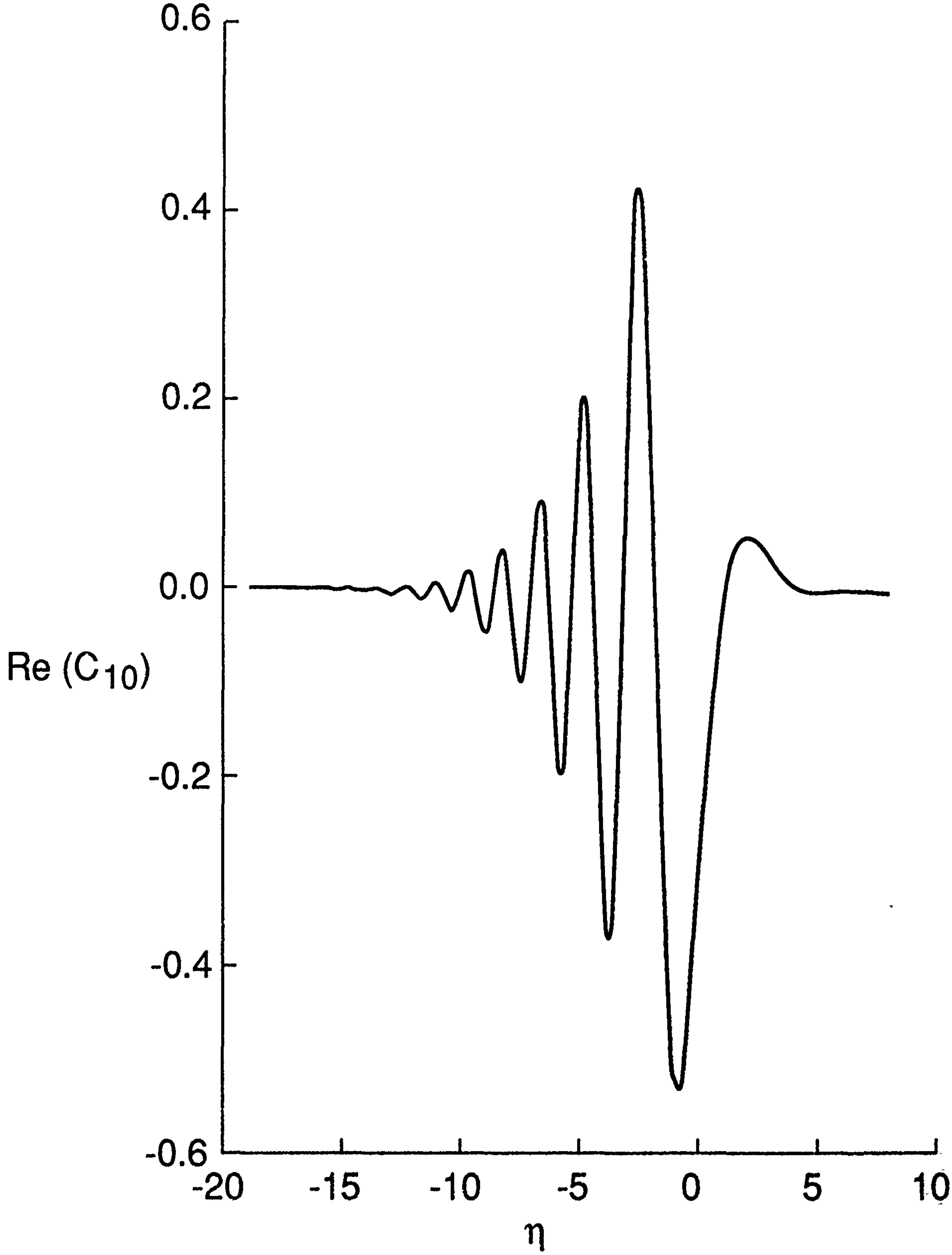
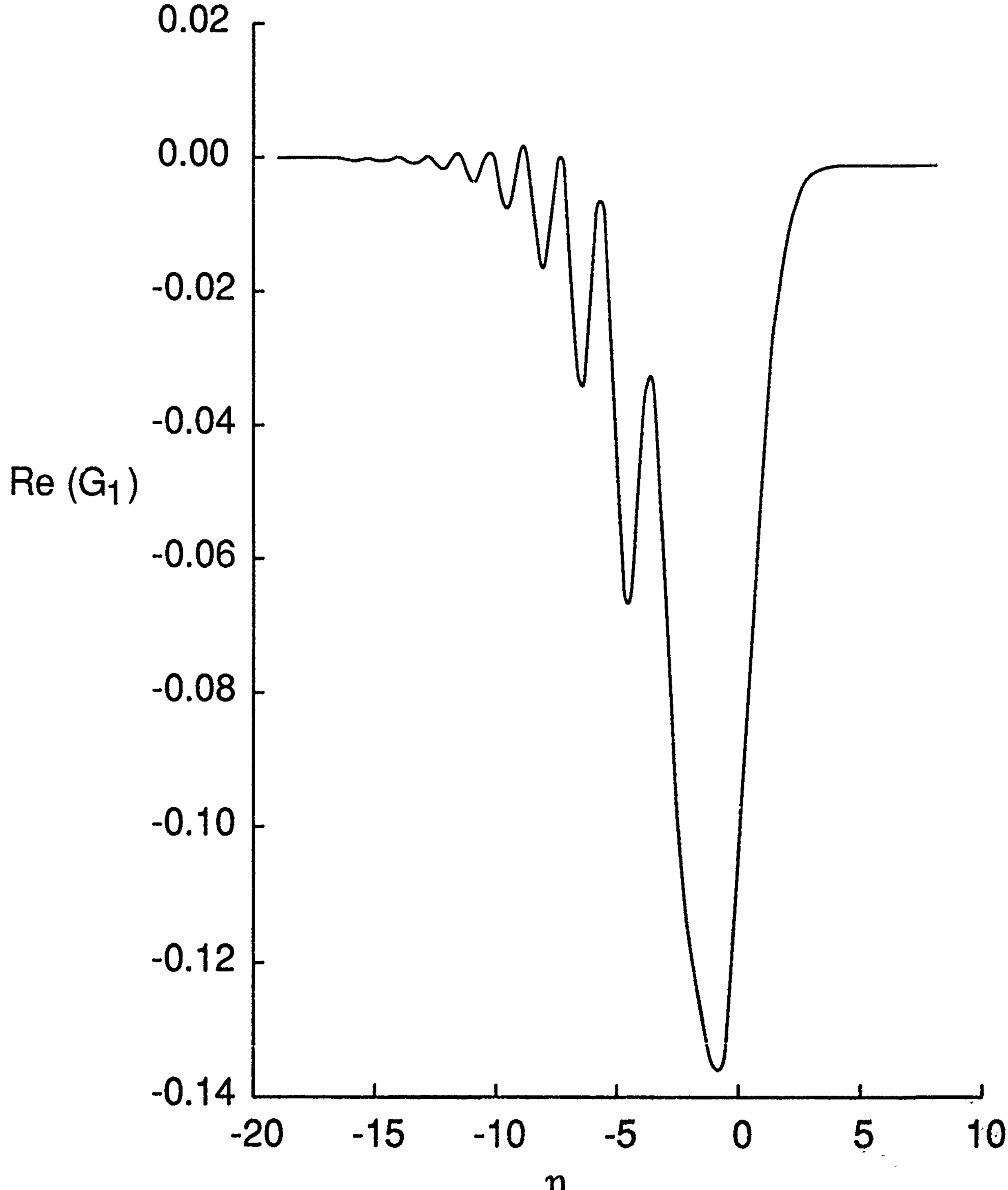


Figure 2c



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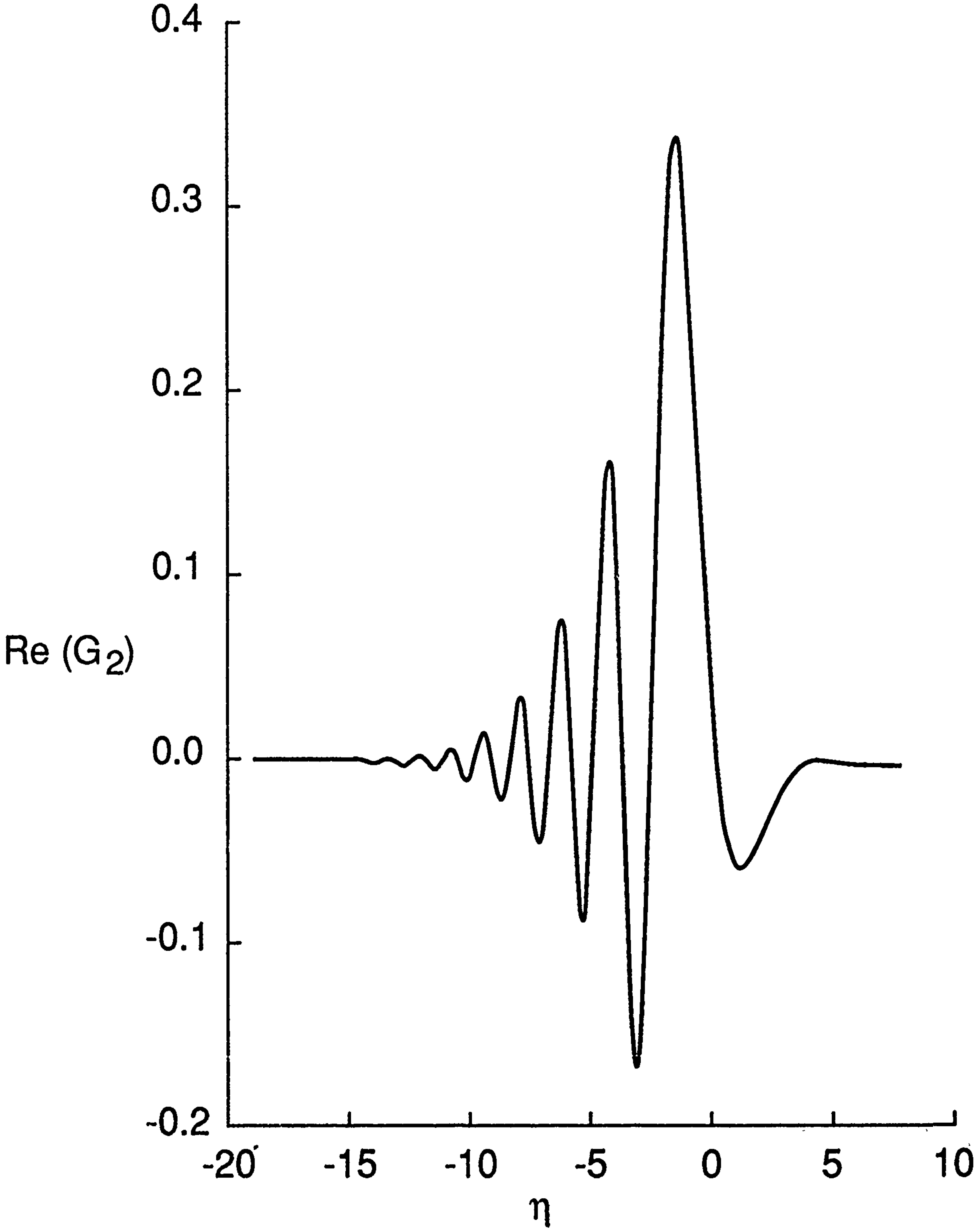

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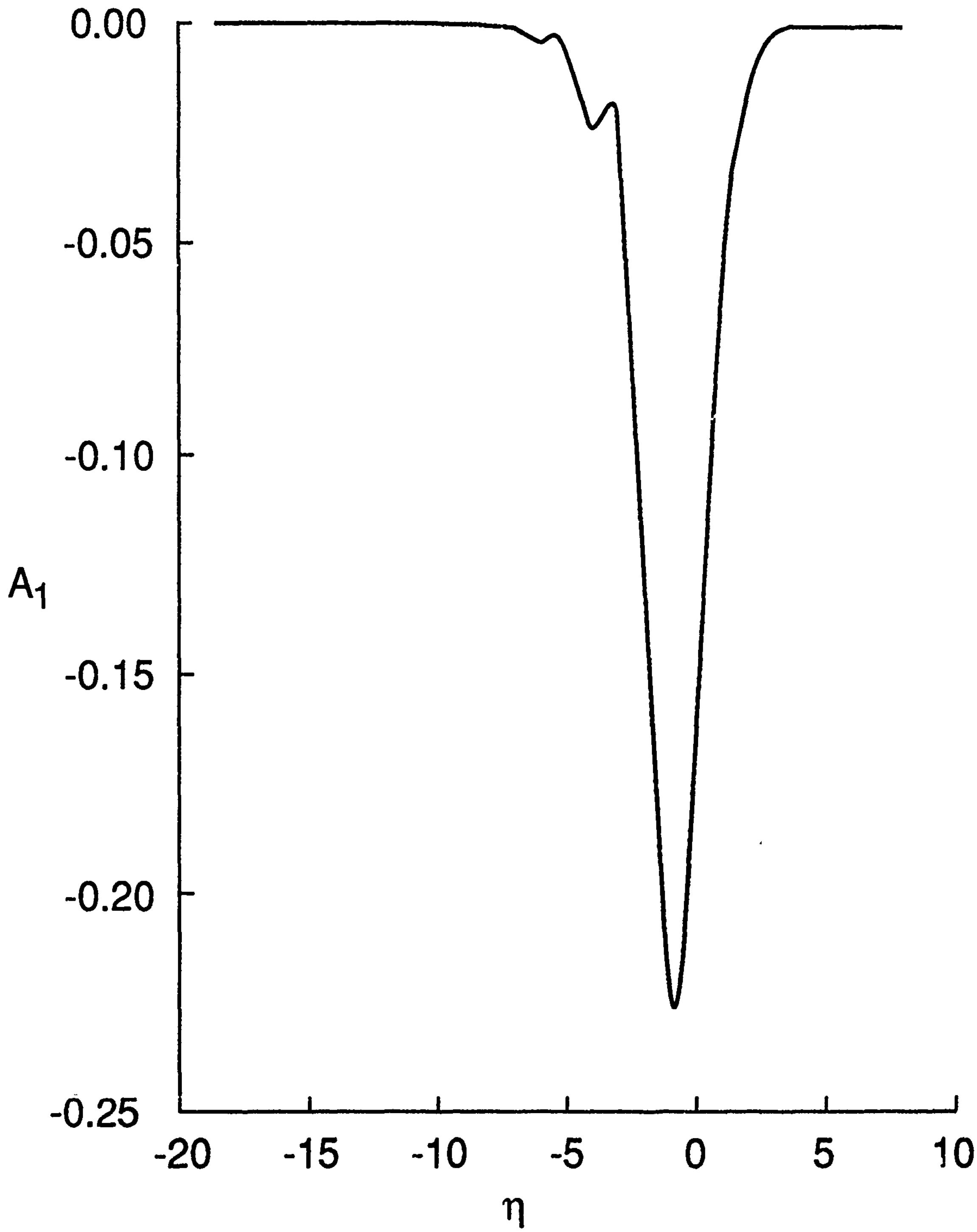

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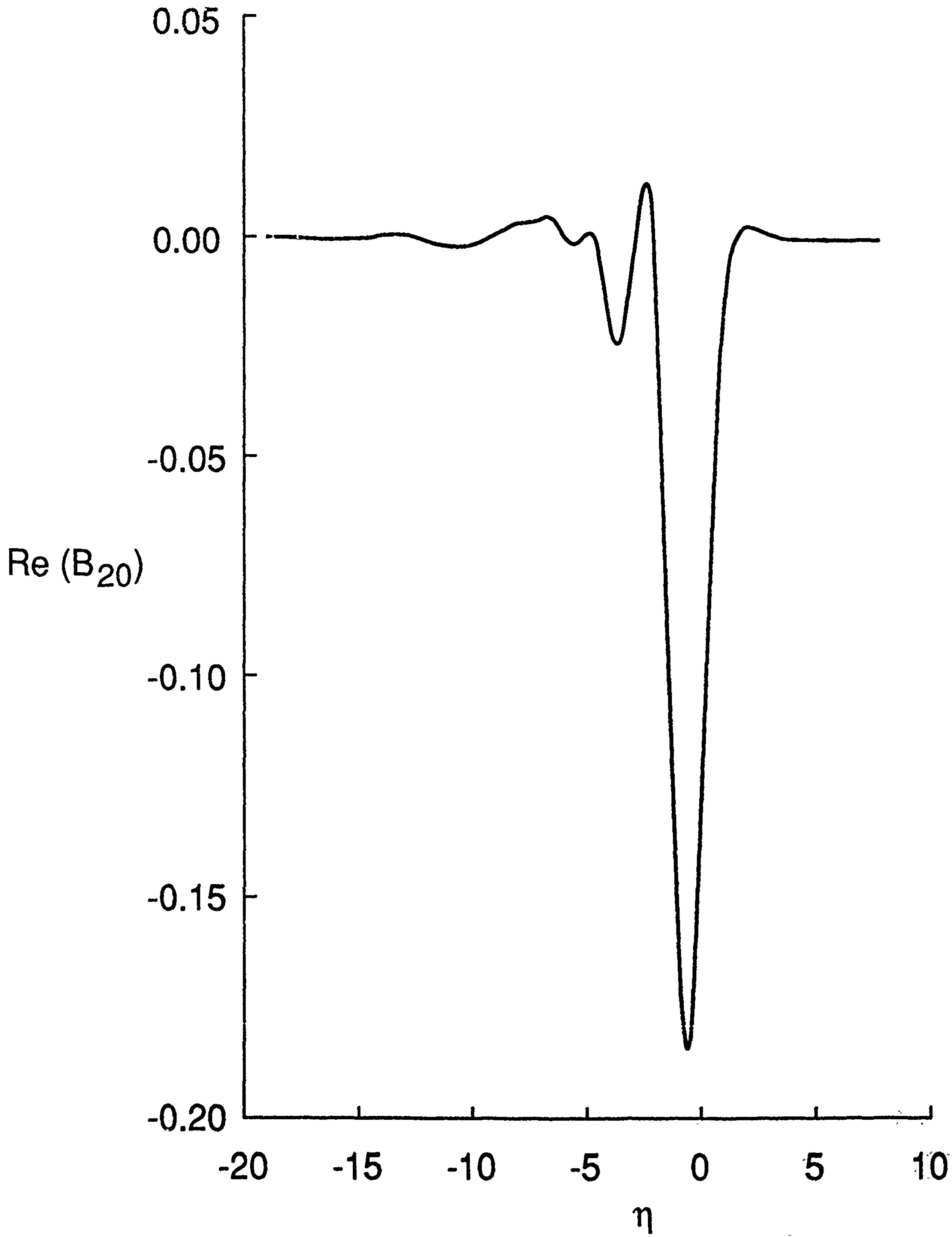

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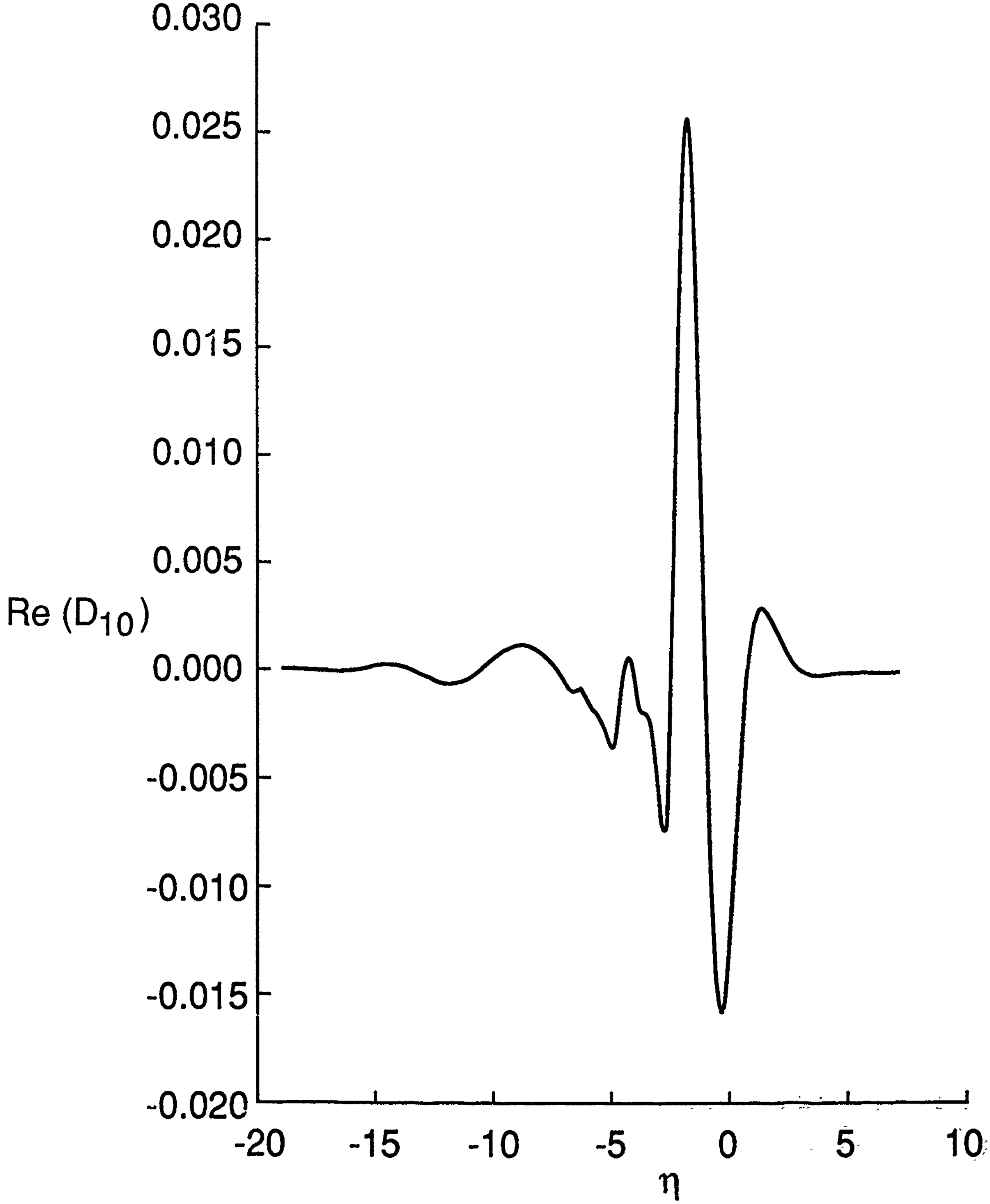
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0.025
0.020
0.015
0.010
0.005
Re (D_{10})
0.000
-0.005
-0.010
-0.015
-0.020
-20
-15
-10
-5
0
5
10
η

Figure 3a

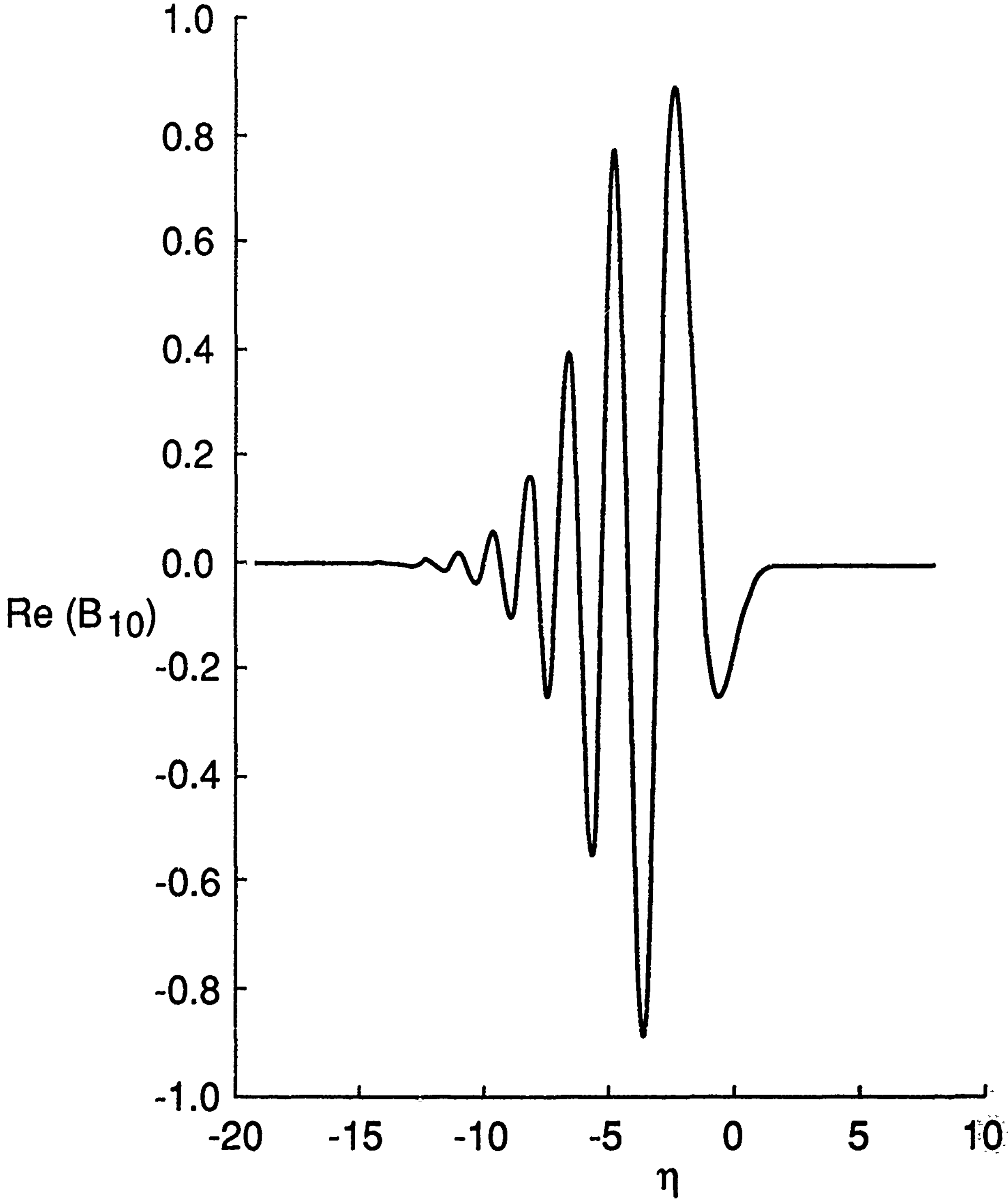


Figure 3b

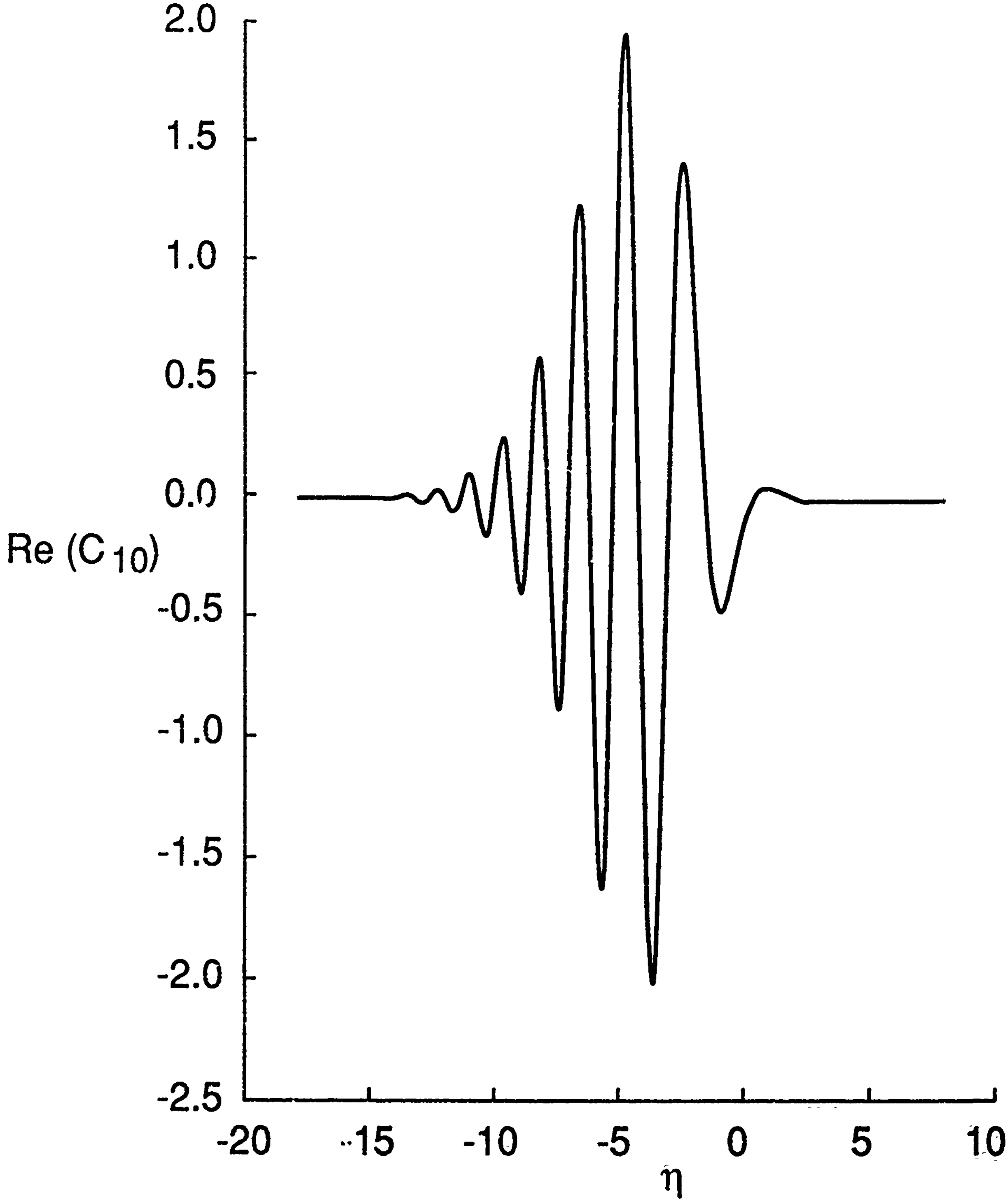
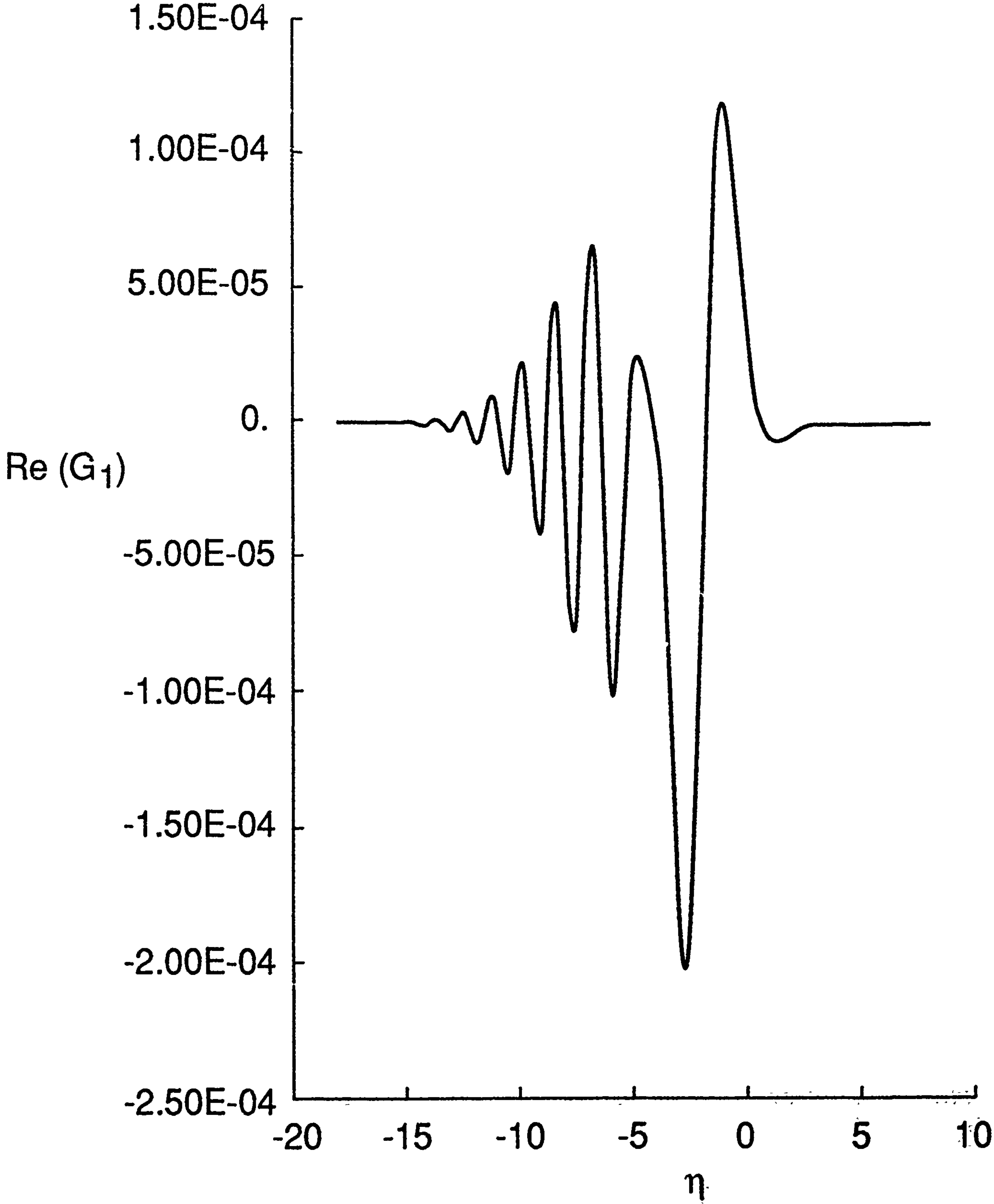
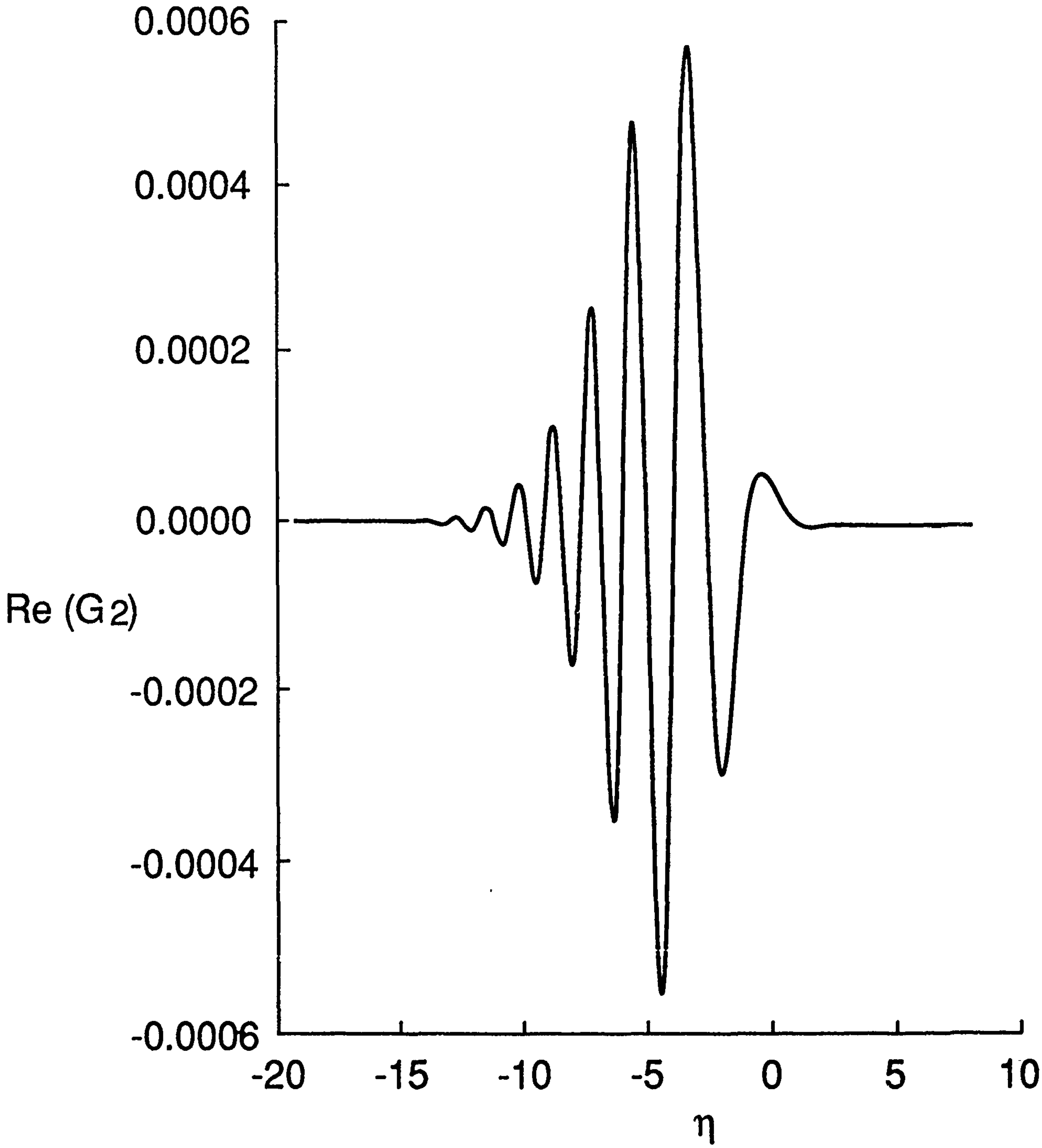


Figure 3c



1.50E-04
1.00E-04
5.00E-05
0.
Re (G1)
-5.00E-05
-1.00E-04
-1.50E-04
-2.00E-04
-2.50E-04
-20
-15
-10
-5
0
5
10
η

Figure 3d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0.0004
0.0002
0.0000
Re (G2)
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-0.0004
-0.0006
-20
-15
-10
-5
0
5
10
η

Figure 3e

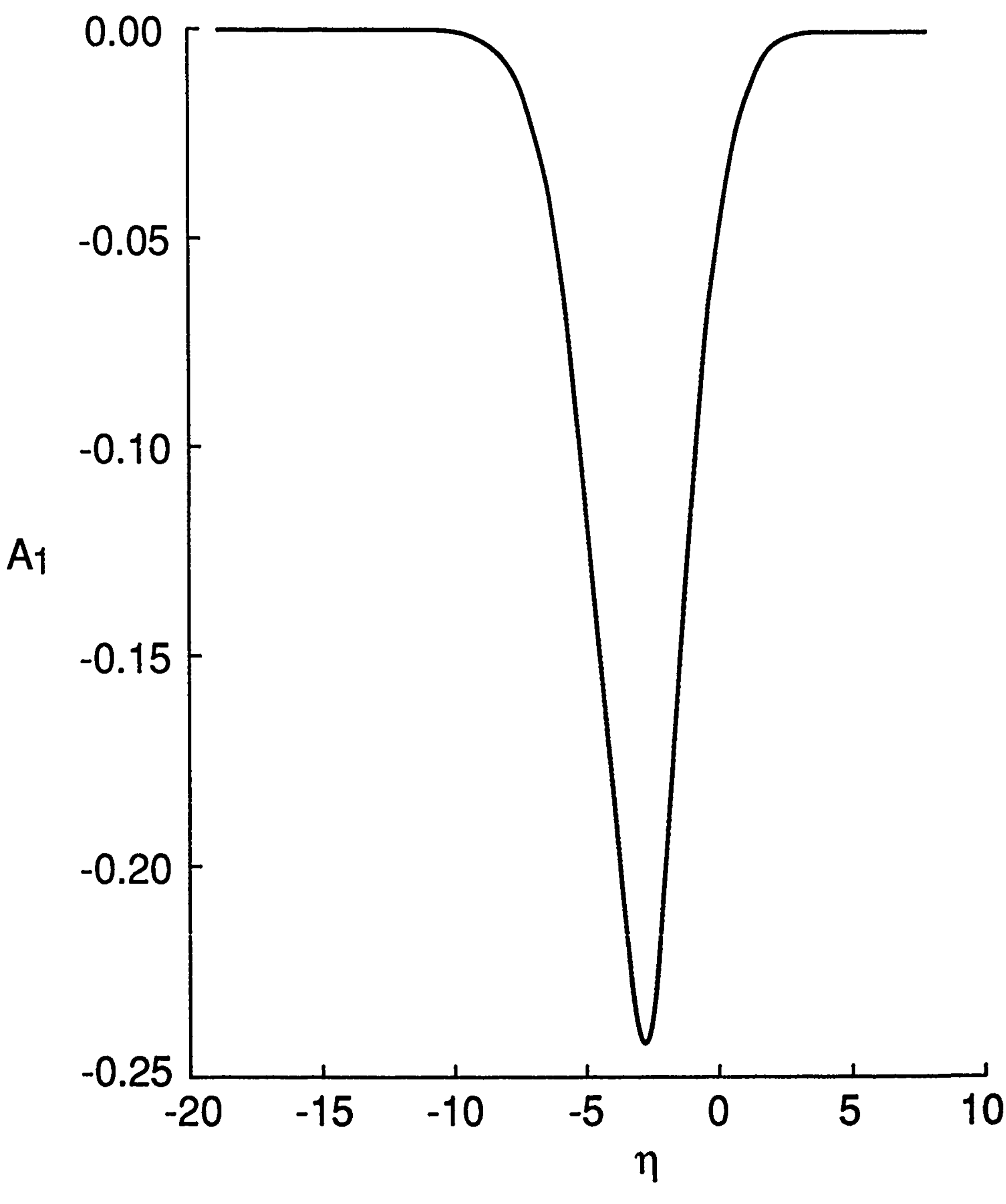

Figure 3f

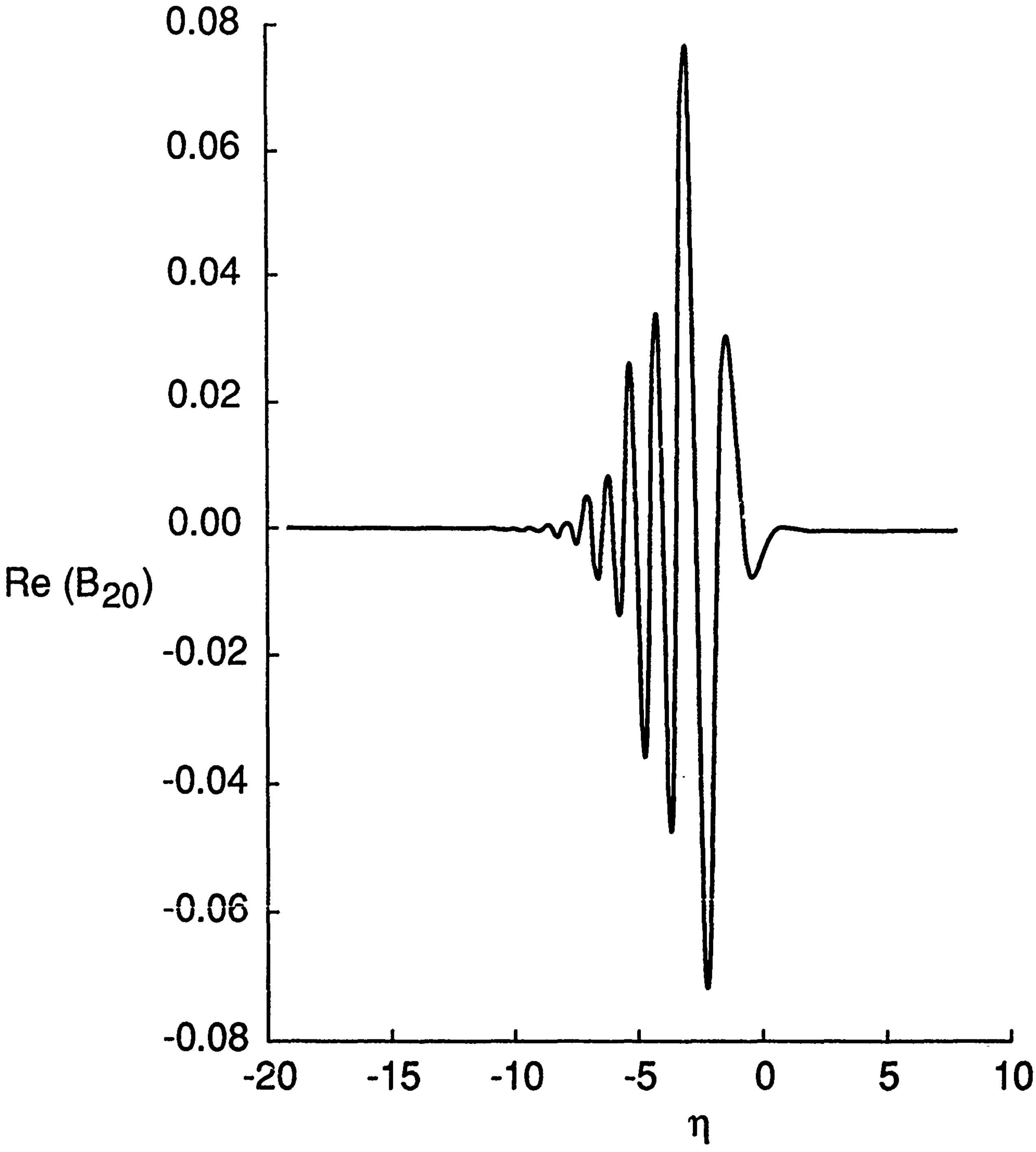

Figure 3g