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MIXED STRATEGIES FOR DYNAMIC GAMES

Louis Carl Westphal, III

California University

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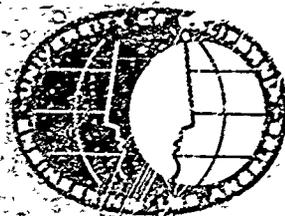
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# MIXED STRATEGIES FOR DYNAMIC GAMES

L.C. WESTPHAL, III

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13. ABSTRACT Development of a comprehensive theory of mathematical games has been hampered by philosophical, conceptual, and practical difficulties. For dynamic games in particular, solution methods are elusive, and algorithms are rare. This is especially apparent for games which require that the competitors randomize, or mix, their tactics even though such randomization is a common property of actual competitive situations. This dissertation is therefore concerned with the development of a technique for the synthesis of mixed strategy solutions of games. A special class of dynamic games is studied: two-person zero-sum noise-free multistage games of fixed duration for which the payoff and dynamic functions are multivariable polynomials and the control vectors are elements of compact hypercubes. The problem is formulated such that known results concerning existence of saddlepoint solutions are applicable; emphasis is on the determination of the value and of the optimal mixed strategies and on the properties of the solution functions. This is achieved by extending and applying the method of dual cones such that the game becomes a maximization problem and the optimal strategies are derived from the interaction of two special convex sets. It is shown that this maximization problem can be approximated in a straightforward and intuitively satisfying manner by a linear programming problem. In the approach used, the state vector of the game is a parameter. For this reason the continuity properties of the functional dependence of the value and the strategies upon this parameter are investigated. One result is that for a game with quadratic payoff and linear dynamics the value function is piecewise quadratic.			

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Louis Carl Westphal III

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School of Engineering and Applied Science  
University of California  
Los Angeles

## PREFACE

Because U.S. Air Force systems be they missile, space, tactical, aeronautical, or other systems inevitably are to be utilized in competitive (differential game) situations and because a comprehensive theory of mathematical games has yet to be developed the results presented in this report were evolved with this goal in mind. Numerous basic results are contained herein with the ultimate goal of a comprehensive theory of differential games in mind, and the utility and significance of the results developed herein are illustrated by application to numerous illustrative examples.

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The research described in this report "Mixed Strategies for Dynamic Games," UCLA-ENG-7280, by Louis Carl Westphal III, was carried out under the direction of C.T. Leondes and E.B. Stear, Co-Principal Investigators in the Schools of Engineering in the University of California at Los Angeles and Santa Barbara, respectively.

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## ABSTRACT

Development of a comprehensive theory of mathematical games has been hampered by philosophical, conceptual, and practical difficulties. For dynamic games in particular, solution methods are elusive, and algorithms are rare. This is especially apparent for games which require that the competitors randomize, or mix, their tactics even though such randomization is a common property of actual competitive situations. This dissertation is therefore concerned with the development of a technique for the synthesis of mixed strategy solutions of games.

A special class of dynamic games is studied: two-person zero-sum noise-free multistage games of fixed duration for which the payoff and dynamic functions are multivariable polynomials and the control vectors are elements of compact hypercubes. The problem is formulated such that known results concerning existence of saddlepoint solutions are applicable; emphasis is on the determination of the value and of the optimal mixed strategies and on the properties of the solution functions. This is achieved by extending and applying the method of dual cones such that the game becomes a maximization problem and the optimal strategies are derived from the interaction of two special convex sets. It is shown that this maximization problem can be approximated in a straightforward and intuitively satisfying manner by a linear programming problem.

In the approach used, the state vector of the game is a parameter. For this reason the continuity properties of the functional dependence of the value and the strategies upon this

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parameter are investigated. One result is that for a game with quadratic payoff and linear dynamics the value function is piecewise quadratic.

Computational aspects of the solution technique are extensively discussed, and several illustrative examples are given to demonstrate various points, including the fact that the principle of optimality cannot always be used with the method of dual cones to find exact solutions to multistage polynomial games. A brief formal discussion of the extension of the method to differential games is also presented.

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## CHAPTER 1

### INTRODUCTION

The mathematical theory of games is still a relatively immature discipline with a multitude of theoretical and practical problems. Solution of those problems will bring about increased understanding of cooperation and competition in such diverse fields as anthropology, economics, military defense, diplomacy, sports, and behavioral psychology. It is even possible that game theory will become a major branch of applied mathematics, for it encompasses optimization theory as a special case while introducing new questions due to its concern with the interactions of multiple intelligent participants.

One objective in the theory of games is to determine, for any given situation, the best tactics for each participant to use and the payoff to each when all use their best tactics. In practice the theory is applied to a mathematical representation, or model, of the actual situation, and the adequacy of a particular analysis depends upon both the sensibility of the model and the intuitive acceptability of the results. This need for realism leads to a requirement that the theory be applicable, for example, to dynamic situations with multiple competitors whose knowledge of the true situation may at times be incomplete, and indeed researchers are attempting to resolve the mathematical difficulties presented by such cases.

It is well known, however, that in many types of competition the participants diversify their tactics so that under similar circumstances their actions vary and are unpredictable to their opponents. Such mixing, or randomization, of tactics is common in many sports and parlor games and in guerrilla warfare. It also underlies such maneuvers as bluffing and feinting. Thus one would expect the theory to produce randomized tactics as solutions of its models.

Surprisingly, although solutions of games based upon static situations are often randomized, this is not presently the case for most dynamic games. Therefore, this paper is concerned with developing a theory which produces randomized tactics as needed in the solution of a particular class of dynamic games. The class studied is that of perfectly competitive situations with only two participants, the so-called two-person zero-sum games. The dynamics of the game are modeled by multistage equations, and each player knows all pertinent information concerning the game except the future tactics of his opponent. The dynamics and payoff functions which define the game are multivariable polynomials. Finally, each play of the game lasts a fixed number of stages, and the players choose their control actions as elements from compact hypercubes.

Such specialized games should prove to have wide application. The two-person zero-sum model is often used, and multistage dynamics may be more accurate for representing applications such as business decisions than continuous dynamics. Furthermore, polynomials are frequently employed in models of real situations.

Particular applications which may be foreseen include pursuit-evasion and weapon allocation problems for defense purposes, optimal pricing and advertising determination for direct business competitions, resource allocation for political campaigns, and perhaps even game plan determination for some sports and parlor games. The theoretical results of this paper will allow approximate solutions of these and other problems for which suitable models of the requisite type are derivable.

The existence of saddlepoint solutions using mixed strategies has been established for this class of problems, the concern in this report is with developing a theory for actual synthesis of those solutions, a task accomplished by extending the theory of dual cones originally developed by S. Karlin, M. Dresher, and L. Shapley for a restricted class of static games. Suitable background material and relevant definitions are in Chapter 2. The theoretical development begins in Chapter 3 with a precise definition of the problem of interest.

The principal theoretical results and discussions concerning approximate solutions are in Chapters 4 and 5. In the first of these, the problem is attacked by solving a special static game. The solution is obtained by reducing the problem of finding the optimal mixed strategies to a problem of determining the generalized moments of such strategies. Next the sets of admissible moments and certain convex cones which they generate are described. The value of the game and the optimal moments are then obtained by exploiting features of the dual convex cones. The chapter is

concluded with discussions of computational aspects of the solution method, including an approximate formulation as a linear programming problem.

In Chapter 5 the effects of introducing the dynamic aspects of the game are examined. The essence of the approach is that the dynamic game is reduced to a sequence of static games in which the system state is a parameter. The applicability of the method of dual cones to finding open-loop and closed-loop optimal strategies is discussed. Then continuity properties of the value and of the optimal strategies as functions of the state vector are evaluated in detail. Finally, the dual cone approach is utilized to prove that the value of games with linear dynamics and quadratic payoff is piecewise quadratic.

Chapter 6 is devoted to four examples which illustrate various aspects of the theory developed in Chapters 4 and 5. Chapter 7 contains a brief, formal discussion of the extension of the methods developed in this report to differential games. A summary of results and a look to the future comprise the concluding chapter, Chapter 8.

The original contributions of this work are embodied in the extension of the method of dual cones to include vector control elements, the creation of a solution technique based upon that method, the manner of formulating the approximate problem so that linear programming may be applied, and certain aspects of the use of the method for multistage games. Among the last of these,

the proof that certain linear-quadratic games have piecewise-quadratic value functions is original, as are portions of the arguments concerning continuity of the optimal mixed strategies. The discussion of the extension to differential games also contains original elements.

## CHAPTER 2

### BACKGROUND

Hundreds of research works concerning various aspects of game theory have been published since the field was founded by von Neumann and Morgenstern in 1944 [1]. In this chapter we review the history and the commonly-used definitions for the control systems-oriented branch of mathematical game theory to which the present study belongs. Section 2.3 contains a survey of the literature which is particularly relevant to the synthesis of mixed strategies for dynamic games.

#### 2.1 TERMINOLOGY

Useful insight into a situation can often be obtained simply by reviewing its terminology. This is definitely the case with game theory. Thus it is fruitful to consider definitions and concepts at this point. This terminology is relatively standard for the field, and we shall neither probe its nuances nor attempt to compile a dictionary.

A game is the complete set of rules, definitions, constraints, goals, etc., which describe a multi-participant interaction, whether it be competitive or cooperative. The participants are called players, and if there are  $n$  such participants, the game is called an  $n$ -player or  $n$ -person game. A single contest or realization of the game is called a play or partie.

In a non-trivial game, the players are able to affect its course and outcome. Mathematically it is said that the  $j^{\text{th}}$  player does this by choosing a control or control vector  $u_j$ , or by choosing a sequence

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$\underline{u}_j = \{u_j\}$  or a time history  $\underline{u}_j = u_j(t)$  of such vectors. Ordinarily the control vectors are chosen from some set  $U_j$ , called the set of admissible controls.

To further his own best interests during a partie, a player does not usually behave haphazardly. Instead he uses a strategy, or set of rules which govern his choice of controls depending upon his observations of the course of the partie. Thus a strategy might be thought of as a mapping  $\psi$  from the set of all possible observed situations into the set of admissible controls. If the control implied by a strategy is always a unique function of the situation, then the strategy is called a pure strategy. On the other hand, if the rule assigns control vectors to a situation in a manner which involves randomness, then it is called a mixed or randomized strategy. The essence of a mixed strategy is the relative frequency of utilization of various control vectors rather than the randomization mechanism, and it is therefore common to refer to probability measures defined on the sets of admissible controls as mixed strategies. Controls with nonzero probability measure in a given situation are the ones which are candidates for utilization, and these are said to belong to the spectrum of the mixed strategy. Note that control vectors chosen using mixed strategies are random variables.

Some games operate within a framework or system which evolves over time (or some other parameter) in a manner which is important to the structure of the game. We call such games dynamic games, and their complement we call static games. The dynamic system is usually described mathematically using a state or state.

vector  $\underline{z}$  which is a function of the controls and of other parameters.

The progression of the state during a partie is described by a dynamics equation, which may be a differential equation,

$$\dot{\underline{z}} = \underline{f}(\underline{z}, \underline{u}_1, \underline{u}_2, \dots, \underline{u}_n; t) \quad (2.1)$$

or a difference equation

$$\underline{z}(i+1) = \underline{f}(\underline{z}(i), \underline{u}_1(i), \dots, \underline{u}_n(i); i) \quad (2.2)$$

In the former case the game is called a differential game, and in the latter it is referred to as a difference game, a discrete differential game, or a multistage game. A dynamic game whose rules prescribe that a partie proceeds for exactly T time units or N stages is called a game of fixed duration.

Along with the direct complications which dynamic games introduce come several conceptual problems. An important one of these is that the nature of strategies must be further refined to account for whether the players are allowed to expect to have knowledge of the state whenever they choose control vectors. If not, then they must consider the possibility of making open-loop control choices when they design their strategies, and the resulting strategies are called open-loop strategies. If the rules allow them to expect that they may always have up-to-date observations on which to base their control choices, then they may design closed-loop strategies which depend upon those observations. For example, in one simple differential game the  $i^{\text{th}}$  player may be required to generate an open-loop mixed strategy function represented by a

conditional cumulative distribution function  $F_i(\underline{u}_i(t) | \underline{z}(\tau), \underline{u}_i(s); \tau \leq s \leq t)$ , whereas in another such game he may design a closed-loop mixed strategy with c.d.f.  $F_i(\underline{u}_i(t) | \underline{z}(t))$ . Clearly these concepts are generalizations of the ideas of open-loop and closed-loop controls. Note that the strategy type is determined by the rules of the game rather than by the conditions obtained during a partie of that game.

Ultimately, each player in a game strives to achieve some goal. For mathematical games this fact is represented by associating with each player  $j$  a payoff functional, which for each partie assigns to that player a real number  $J_j$  that depends upon the structure of the game and the course of the partie. In particular, if  $\underline{u}_j$ ,  $j=1, 2, \dots, n$ , denotes control histories and  $\underline{z}$  denotes state histories, then we write

$$J_j = J_j(\underline{z}, \underline{u}_1, \underline{u}_2, \dots, \underline{u}_n) \quad j=1, 2, \dots, n \quad (2.3)$$

to represent the payoffs. A game for which  $\sum_{j=1}^n J_j = 0$  is called a zero-sum game; any other game is nonzero-sum. Depending upon the nature of the game, the payoffs may belong to finite or infinite sets and may be bounded or unbounded.

Each player in a game chooses his control history during a partie, and thus designs his strategy, to best serve his own interests. The exact nature of "best" is dependent upon the rules of the game; for example, a player may in some games submerge his direct interests to those of a group and in other games may strive for maximum security of payoff rather than to maximum payoff. Furthermore, frequently a function of the payoff such as its mean is extremized rather than the raw payoff. In any case, if it is possible

for each player to design a strategy which in the game sense best serves his interests in terms of a function  $f_j$  of the payoffs, then his payoff when all players use such optimal strategies is called the value of the game to him. We write this as

$$w_j = \text{val} (f_j (J_1, J_2, \dots, J_n)) \quad j=1, 2, \dots, n \quad (2.4)$$

Because the exact nature of the maximization is so intimately related to the particular structure of a game, it is generally difficult to be more definitive than this except for one particular class of problems, the class of two-person zero-sum games.

Two-person zero-sum games are the subject of intense research interest and accordingly are the source of considerable specialized terminology. In such games, it is possible to define a single payoff function  $J$  which has the property that

$$J = J_1 = - J_2 \quad (2.5)$$

Such games are often called perfectly competitive, since by their nature one player's gain is the other's loss. In these games a rational player attempts to maximize his minimum possible expected payoff; i. e., Player I attempts to maximize the minimum possible  $f(J)$  and Player II tries to minimize the maximum of  $f(J)$ . If we call  $\Psi_i$  the strategy sets for the players,  $i=1, 2$ , then we write

$$\hat{J}_1 = \max_{\psi_1 \in \Psi_1} \min_{\psi_2 \in \Psi_2} [f(J)]$$

$$\hat{J}_2 = \min_{\psi_2 \in \Psi_2} \max_{\psi_1 \in \Psi_1} [\mathcal{F}(J)]$$

as the goals of the two players. If  $\hat{J}_1 = \hat{J}_2$ , then this common number is by definition the value of the game. It is clear that

$$w = \hat{J}_1 = \hat{J}_2 = \mathcal{F}(J) \Big|_{\psi_1^0, \psi_2^0} = \text{val } \mathcal{F}_1(J_1, J_2)$$

has the property

$$\mathcal{F}(J) \Big|_{\psi_1, \psi_2^0} \leq w \leq \mathcal{F}(J) \Big|_{\psi_1^0, \psi_2} \quad (2.6)$$

where the notation indicates that the payoff function is to be evaluated using the optimal strategy  $\psi_i^0 \in \Psi_i$  and any admissible strategy  $\psi_j \in \Psi_j$ ,  $j \neq i$ . Condition (2.6) is called a saddlepoint condition, and a strategy  $\psi_i^0$  which yields this condition is called an optimal strategy, a saddlepoint strategy, or a mini-max strategy. These notions are also used in some other classes of games.

If at least one player in a game lacks some essential piece of information, such as exact knowledge of the state vector, the nature of the dynamics, or the payoff for some player, then the game is called a game of imperfect information, or a stochastic game; otherwise, the game is one of perfect information. Common dynamic games of imperfect information are those for which at least one player has knowledge of a vector function of the state,

$$y_i = y_i(\underline{z}, \underline{w}) \quad (2.7)$$

where  $\underline{w}$  is a random vector, rather than of the state vector  $\underline{z}$ , or where the dynamics functions depend on a random vector  $\underline{\zeta}$  as well as on the state and the controls. Many games do not fall naturally into either category, and their precise classification must be by convention. We shall use the following convention: if the controls and state are random variables due solely to the use of mixed strategies and the participants have equivalent knowledge of the game, then we shall call it a game of perfect information.

With the above concepts in mind, we are able to characterize a great variety of mathematical games. In this report are described the optimal mixed strategies for two-person zero-sum multistage games with fixed duration and perfect information and with payoff and dynamics functions characterized by polynomials. Both open-loop and closed-loop strategies are examined.

## 2.2 THE HISTORY OF GAME THEORY

A great amount of research concerning mathematical game theory has been published: A bibliography compiled in 1959 [2] has more than one thousand entries, and a recent bibliography of differential games [3] contains over two hundred references and is still incomplete. Therefore, any overview of the field is useful but necessarily cursory. This section reviews the history of the branch of game theory which is most closely related to this report.

Although there are earlier relevant publications, it is generally conceded that game theory had its birth with the publication of the classic work of von Neumann and Morgenstern [1]. Besides creating the field, these researchers contributed some standard

results, the most important being a theorem proving that for static two-person zero-sum games for which the controls must be chosen from finite sets, optimal strategies and a value would exist provided that mixed strategies were allowed. This was later proven in alternative ways, among them the dual theory of linear programming, and it was shown that the mixed strategies could be computed using linear programming (See, for example, Gass[4]).

Following the publication of that book, game theory was the subject of intensive research interest for several years. Interest in static games was particularly high, and among the results are algorithms for solving general two-person zero-sum games with finite control sets and theorems showing that a value and optimal mixed strategies exist for certain two-person zero-sum games with infinite control sets. The former fact was alluded to in the preceding paragraph, and initial versions of the latter are attributed by Kuhn and Tucker [5], among others, to J. Ville and to A. Wald. Blackwell and Girshick [6] supply a fairly comprehensive discussion of the mini-max theorem.

Along with these general results, many special two-person zero-sum games were examined, including in particular the so-called games over the unit square, in which the players choose controls as real numbers from the unit interval  $[0, 1]$  and the payoff functions are of special forms, such as polynomials or convex functions. An excellent source for this period, with interesting and enlightening commentary by the editors, is the series Contributions to the Theory of Games [5], [7], [8], [9].

A new dimension was added to game problems in the middle 1950's by Isaacs when he created dynamic games, particularly two-person zero-sum differential games [10], [11], [12], [13]. His highly original work is available as a book [14] which is best read along with a book review by Ho [15]. In brief, Isaacs is concerned with examples of problems with dynamics

$$\dot{\underline{z}} = \underline{f}(\underline{z}, \underline{u}, \underline{v}; t) \quad \underline{z}(0) = \underline{z}_0$$

and payoff function

$$J(\underline{z}_0, \underline{u}(t), \underline{v}(t)) = g_f(\underline{z}_c, t_c) + \int_0^{t_c} g(\underline{z}(t), \underline{u}(t), \underline{v}(t), t) dt \quad (2.8)$$

where  $t_c$  is the time at which a given terminal manifold is reached and  $\underline{z}_c$  is the final position on that manifold. He assumes that the payoff has a saddlepoint when pure strategies are used and argues that if the value function  $J^*(\underline{z}, t)$  exists, it satisfies his Main Equation One, or  $ME_1$ ,

$$\frac{\partial J^*}{\partial t} + \min_v \max_u \left[ (\nabla_{\underline{z}} J^*)^T \underline{f}(\underline{z}, \underline{u}, \underline{v}, t) + g(\underline{z}, \underline{u}, \underline{v}, t) \right] = 0 \quad (2.9)$$

where  $\nabla$  is the gradient operator. To find this, he applies what he calls the Tenet of Transition, a game theory analog of Bellman's Principle of Optimality which he apparently found independently. In principle, (2.9) may be solved for  $\underline{u}^0 = \underline{u}^0(\underline{z}, \nabla_{\underline{z}} J^*, t)$  and  $\underline{v}^0 = \underline{v}^0(\underline{z}, \nabla_{\underline{z}} J^*, t)$ , which are then inserted to give the Main Equation Two, or  $ME_2$ ,

$$\begin{aligned} \frac{\partial J^*}{\partial t} + (\nabla_{\underline{z}} J^*)^T \underline{f}(\underline{z}, \underline{u}^0(\underline{z}, \nabla_{\underline{z}} J^*, t), \underline{v}^0(\underline{z}, \nabla_{\underline{z}} J^*, t), t) \\ + g(\underline{z}, \underline{v}^0(\underline{z}, \nabla_{\underline{z}} J^*, t), \underline{v}^0(\underline{z}, \nabla_{\underline{z}} J^*, t), t) = 0 \end{aligned} \quad (2.10)$$

This equation is of Hamilton-Jacobi type, and is commonly referred to as such. Equation (2.9) is often called a Hamilton-Jacobi-Bellman equation or a pre-Hamiltonian equation.

Using his main equations, Isaacs also contributes a sufficiency theorem. In essence, he finds that if  $J^*(\underline{z}, t)$  is a unique continuous function satisfying the ME's and the boundary condition  $J^*(\underline{z}_c, t_c) = g_f(\underline{z}_c, t_c)$ , then  $J^*$  is the value  $w(\underline{z}, t)$  of the game and any pure strategies which furnish the min-max in (2.9) and cause the desired end point to be reached are optimal. This is true in a limiting sense, that is, as the limit of a convergent series of discrete approximations to the differential game.

Interest in differential games built up gradually for several years and culminated in a major work by Berkovitz [16], who extended results of the classical calculus of variations to zero-sum two-person differential games. His principal results are that under some fairly restrictive conditions, the Hamiltonian-like function

$$H(\underline{z}, \underline{u}, \underline{v}, \underline{p}) = \underline{p}^T \underline{f}(\underline{z}, \underline{u}, \underline{v}) + g(\underline{z}, \underline{u}, \underline{v}) \quad (2.11)$$

satisfies, for optimal controls  $\underline{u}^0$  and  $\underline{v}^0$ ,

$$\dot{\underline{z}} = \nabla_{\underline{p}} H(\underline{z}, \underline{u}^0, \underline{v}^0, \underline{p}) \quad (2.12)$$

$$\dot{\underline{p}} = -\nabla_{\underline{z}} H(\underline{z}, \underline{u}^0, \underline{v}^0, \underline{p}) \quad (\text{Cont'd})$$

(2.12)

$$\nabla_{\underline{u}}^T H + \underline{\mu}^T \frac{\partial K}{\partial \underline{u}} = 0$$

$$\underline{\mu} \leq 0 \quad \underline{\mu}_i K_i = 0$$

$$\nabla_{\underline{v}}^T H + \bar{\underline{\mu}}^T \frac{\partial \bar{K}}{\partial \underline{v}} = 0$$

$$\bar{\underline{\mu}} \geq 0 \quad \bar{\underline{\mu}}_i \bar{K}_i = 0$$

where  $\underline{K}$  and  $\bar{K}$  are vector constraint functions on  $\underline{u}$  and  $\underline{v}$ , respectively, and  $\underline{\mu}$  and  $\bar{\underline{\mu}}$  are associated multipliers. He also establishes a form of Hamilton-Jacobi equation (2.10) and sufficiency conditions using field concepts. The results apply to problems which have solutions in pure strategies.

Once these basic results were established, a great many researchers applied them to special cases and interpretations, and to extensions of the same class of problems. Among these, a very influential work was contributed by Ho, Bryson, and Baron [17], who studied a particular game with linear dynamics and quadratic payoff which has pure strategy solutions. Other contributions in the same general area of two-person zero-sum differential games include those of Wong [18], Meier [19], Meschler [20], and Wu and Li [21]. Interesting geometric work in an augmented state space is found in works by Leitmann and others. Blaquièrre, Gérard, and Leitmann [22] is representative of this approach.

A variation of the above differential game has received attention from several researchers, including some of the prominent Russians. If the payoff for a two-person zero-sum game is the time  $T$  of attaining a terminal manifold, a problem is created which may not end; i. e., it may be that  $T = \infty$ . Pontryagin [23] shows that if an optimal payoff exists, his maximum principle may be applied to such

games provided that the Hamiltonian can be written

$$\begin{aligned} H(\underline{z}, \underline{u}, \underline{v}, \underline{p}) &= p_0 + \underline{p}^T f(\underline{z}, \underline{u}, \underline{v}) \\ &= H_1(\underline{z}, \underline{u}, \underline{p}) + H_2(\underline{z}, \underline{v}, \underline{p}) \end{aligned} \tag{2.13}$$

Other results for related problems appear in such works as Chatopadhyay [24] and Varaiya [25].

Research interest is now shifting to games other than two-person zero-sum differential games of perfect information which have pure strategy optimal solutions. In particular, dynamic games with  $n$  players, with imperfect information, or with mixed strategy solutions are being investigated. These areas overlap, of course, but it is enlightening to consider them separately. The third area is surveyed in the next section.

The fundamental philosophical problem of  $n$ -person games and the closely related nonzero-sum games is the definition of what is meant by a solution. There are at least three basic solution types: min-max for each player, equilibrium solutions in which no player can improve his payoff unilaterally, and bargaining solutions in which no player can change his strategy without adversely affecting at least one other player. Therefore, the rules of the game, and particularly questions of agreements and side payments among players, dominate the theory. References [1], [7], [8], and [9] contain some of the relevant publications for static games. Case [26] and Starr and Ho [27], [28], who also have published similar works elsewhere, are leaders in studies of the  $n$ -person differential game problem. In particular, they have found that when equilibrium solutions are sought,

individual Hamilton-Jacobi equations apply for each player along the optimum state trajectory and that a method of characteristics is sometimes applicable. Min-max solutions may be found for each player by applying two-person game theory, and bargaining solutions are related to optimal control problems with vector payoff functions.

Studies of games with imperfect information have generally been concentrated on two-person zero-sum dynamic games with noisy state transition and noisy observations of the state by the players. The fundamental problem is that a player must base his controls on his available information, which tends to be incomplete and inexact, and must guess not only the state, but what his opponent thinks the state is, what his opponent thinks he thinks the state is, ad infinitum. The payoff is usually taken as the mean of the given payoff function.

Behn and Ho [29] circumvent some of the computational problems by assuming a control form and then determining its parameters based upon the statistics of the noise processes. Rhodes and Luenberger [30], [31] show that a type of stochastic Hamilton-Jacobi-Bellmann approach is applicable when the contenders are able to determine their opponent's strategy, and it is noteworthy that their results do not require pure strategies. An interesting approach is suggested by Sugino [32], who postulates bounded noise and thus is able to find mini-max strategies by using regions of attainability. Other important research includes that of Kushner and Chamberlain, who in several works, among them [33], study the Markov process characteristics of stochastic games, and Bley and Stear [34], who use a Bayesian analysis of multistage games to find conditions

for pure strategies.

In closing this section, we remark that there is much to be done even in the fields so far considered. It is noteworthy that much of the work on dynamic games since Isaacs has been so highly control system oriented that it has led to what has only recently been recognized as a distortion of the approach and a lack of recognition of some of the peculiar, fascinating properties which mathematical games possess. This fact has been noted by Isaacs [35] and Ho [36], for instance.

### 2.3 THE SYNTHESIS OF RANDOMIZED STRATEGIES FOR DYNAMIC GAMES

Early researchers actively sought mixed strategy solutions to their static problems. We have already noted that linear programming yields mixed strategies for two-person zero-sum static games with finite control sets. Other games, such as games over the unit square, that is, games for which the controls are scalars chosen from the unit interval  $[0, 1]$ , were examined, and solutions were discussed for two-player zero-sum games for which the payoffs are convex functions (Bohnenblust, Karlin, and Shapley [37]), polynomial functions (Dresher, Karlin, and Shapley [38]), and bell-shaped functions (Karlin [39]), among others. Many of the results from this era may be found in Karlin's book [40].

The research of Karlin, et al, [38], [40], and [41], on polynomial and separable games is particularly relevant to our problem. They, however, are concerned solely with static games with scalar controls. They show that for games with separable payoff functions the problem of finding optimal mixed strategies can be reduced to finding moments of those strategies. The latter problem

is then examined for games of known value using the method of dual convex cones. Their concern is with characterizing the relevant sets, and they consider neither synthesis of solutions using the dual cones, problems with vector controls, nor the effects of introducing dynamics to the game.

Few other researchers have considered extending the theory of static games to dynamic games. Bley [42] suggests the application of the theory of convex games and works a scalar multistage example in his study of linear-quadratic games. Cliff [43], who is generally discouraging about the utility of mixed strategies in realistic dynamic games, suggests analyzing the pre-Hamiltonian using static game theory and examines a simple differential game example using the theory of bell-shaped games. Rhodes [44] employs arguments related to the theories of convex and polynomial static games in examples of linear-quadratic dynamic games. None of these researchers is primarily concerned with synthesizing mixed strategies, and their efforts in this regard are confined to examples.

Techniques other than extensions of static game theory have been suggested. In a series of publications Berkovitz and Dresner [45], [46], [47] evaluate tactical air-war problems which have linear payoff and multistage limited-linear dynamics. Their solutions are obtained by ad hoc methods which do not appear to be of general interest.

An interesting approach suggested by Ho [48] and extended by Speyer [49] is to force the controls to be random variables by introducing a dependence of the controls on random vectors. Speyer

does this by choosing controls of the form

$$\underline{u} = K\underline{\hat{z}} + \underline{\xi} \quad (2.14)$$

where  $\underline{\hat{z}}$  is a state variable estimator and  $\underline{\xi}$  is a white noise vector with zero mean and controllable covariance  $Q$ . His problem, a particular linear-quadratic game, is such that only the statistics of the random variables, rather than their instantaneous values, are of importance, and the problem becomes one of finding the gain  $K$  and covariance matrix  $Q$ . Thus the problem is considerably different in means, if not ends, from that of synthesizing the randomness by generating probability distributions for the controls.

In an interesting and provocative paper, Chattopadhyay [50] points out that since in the game surface approach the normals to the surface are intimately related to the optima' strategies, finite mixed strategies might be related to "mixed normals." Thus one can in principle seek an optimal normal and then relate it to pure normals and to mixtures of pure strategies. As with much of the game surface technique, this appears to be more useful for supplying insight than for construction of solutions.

Another suggestion is made by Sarma, Ragade, and Mandke [51]. Arguing purely formally, they state that the value must satisfy a stochastic Hamilton-Jacobi-Bellman partial differential equation with simultaneous extrema in the probability density functions of the mixed strategies of the two contestants in a zero-sum differential game. Existence or uniqueness of solutions is neither proved nor claimed. Since the concept of probability densities does

not appear to be useful (because they cannot represent pure strategy regions as degenerate cumulate probability distributions can), it is likely that the particular result of Sarma, et al, will have limited application.

Smoliakov [52] formulates the problem slightly differently to find mixed strategies for a two-person zero-sum differential game. By requiring that the dynamics equation hold in a mean sense

$$E_{(\underline{u}, \underline{v})} [\dot{\underline{z}} - \underline{f}(\underline{z}, \underline{u}, \underline{v}, t)] = 0 \quad (2.15)$$

rather than in the absolute sense, he is able to put the problem of mini-maxing the mean of the payoff over the mixed strategies into a form which can be attacked by variational methods. The physical significance of (2.15) is debatable, however.

Little other work concerning actual synthesis of mixed strategies has been performed. Some researchers have been unconcerned with synthesis and neither found nor ruled out mixed strategies. The publications of Rhodes and Luenberger [30], [31] and Rhodes [42] are examples of this.

We have already mentioned that much of Chapter 4 represents extensions of the work of Karlin and others. Another portion of the foundation of our research is the fact that a saddlepoint solution indeed exists for the static and the open-loop problems formulated, for proof of which Blackwell and Girshick [6] is one of many possible references. For the closed-loop dynamic problem, the dynamic programming approach is used. This has been used by a number of authors; its validity for the problems of concern here has been stated as a theorem, for example, by Fleming [53].

## CHAPTER 3

### PROBLEM STATEMENT

This research was motivated by the desire to synthesize solutions for a particular class of mathematical games, although many of the results have a more general domain of applicability than this. The goal may be stated as follows: we seek to find the value and the cumulative probability distributions representing the optimal mixed strategies, both open-loop and closed-loop, for the class of fixed-duration two-person zero-sum multistage games characterized by polynomial dynamics and payoff functions and by noise-free information. This statement is clarified and the importance of such problems is discussed in the following sections.

#### 3.1 SYSTEM SCENARIO

The systems of interest to us are dynamic systems which proceed in a step-wise manner under the influence of simultaneous inputs from two controllers. Thus we are concerned with sequences of real  $l$ -vectors  $\{\underline{z}(i)\}$ ,  $m$ -vectors  $\{\underline{u}(i)\}$ , and  $n$ -vectors  $\{\underline{v}(i)\}$  (where  $i$  is an indexing variable which traverses the real integers) which are interrelated according to the dynamics equation

$$\underline{z}(i+1) = \underline{f}(\underline{z}(i), \underline{u}(i), \underline{v}(i); i) \quad (3.1)$$

The functions  $\underline{f}$  are presumed known to the players and by assumption are polynomial functions of their arguments  $\underline{z}(i)$ ,  $\underline{u}(i)$ , and  $\underline{v}(i)$  and are indexed by the stage index  $i$ . The vectors have the following additional properties for each  $i$ :

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$\underline{z}(i)$  - Belongs to euclidean  $l$ -space  $E^l$ . Called the state or state vector of the system.

$\underline{u}(i)$  - Chosen from a unit hypercube  $U$  in  $E^m$ ,

$$U = \{ \underline{u} \mid u_i \in [0, 1], i=1, 2, \dots, m \},$$

(3.2)

by a rational controller called Player I or the maximizer.

$\underline{v}(i)$  - Chosen from a unit hypercube  $V$  in  $E^n$ ,

$$V = \{ \underline{v} \mid v_i \in [0, 1], i=1, 2, \dots, n \},$$

by a rational controller called Player II or the minimizer.

A game may be described for this system by introducing rules and a payoff function. We are concerned with games such that a particular play, or partie, proceeds from a given initial state  $\underline{z}$ , which is identified with stage 1, i. e.,  $\underline{z}(1) = \underline{z}$ , for a fixed number  $N$  stages. Two variations on the basic rules are of interest.

In the first game, called the game of closed-loop strategies, each controller, cognizant of the state  $\underline{z}(i)$ , of the history of play (i. e., of  $\underline{z}(1), \underline{z}(2), \dots, \underline{z}(i-1), \underline{u}(1), \underline{u}(2), \dots, \underline{u}(i-1), \underline{v}(1), \underline{v}(2), \dots, \underline{v}(i-1)$ ), of the dynamics  $f$  and the payoff function  $J$ , and of the number  $N-i$  of remaining stages, but ignorant of the other controller's future control vector choices, chooses a control vector from his set of admissible controls  $U$  (or  $V$ ). This happens for each  $i, i=1, 2, \dots, N$ ; each participant fully expects it to do so and hence designs closed-loop strategies.

In the second game, called the game of open-loop strategies, the controllers cannot be certain of ever receiving updated data. For this reason they design open-loop controls to use for the remainder of the game, and recompute these if any new data become available. Data are assumed available to both players or to neither; they have equivalent knowledge of the state.

For either of these variations, at the end of the game a scalar amount  $J$  determined by

$$\begin{aligned}
 J &= J(\underline{z}; \underline{u}(1), \underline{u}(2), \dots, \underline{u}(N), \underline{v}(1), \underline{v}(2), \dots, \underline{v}(N)) \\
 &= g_{N+1}(\underline{z}(N+1)) + \sum_{i=1}^N g_i(\underline{z}(i), \underline{u}(i), \underline{v}(i))
 \end{aligned}
 \tag{3.5}$$

is paid by Player II to Player I. The functions  $g_i$ ,  $i=1, 2, \dots, N+1$ , are assumed to be polynomial functions of their arguments.

By describing the dynamics, rules, and payoff function, we have defined a game. The concepts of solutions to this game are pursued in the next section, and the particulars of solutions are treated in Chapters 4 and 5.

### 3.2 THE CONCEPT OF SOLUTION: VALUE FUNCTIONS AND STRATEGIES

The two players in the game of Section 3.1 are presumed to be both intelligent and rational in that each will attempt to optimize the payoff  $J$  according to his own best interests. To ensure his success, each player employs a strategy, which we may think of as a rule or mapping which implies an admissible control vector for each contingency in the game, that is, for each possible position  $\underline{z}$  and stage  $i$ .

If a unique control vector is implied by this function for each contingency, then the function is called a pure strategy. If the mapping also depends on a random variable, so that the selected control depends upon the realized value of this random variable in addition to  $z$  and  $i$ , then the function is called a randomized or mixed strategy. It is clear that a pure strategy is a special case of mixed strategies.

Since finding good strategies for the competitors is fundamental to solving games, we must refine the notion of mixed strategies. The key concept is that at each stage each player chooses his control vector in a (possibly) random manner. The exact means of introducing the randomness is incidental; the crucial factor is the relative frequency of utilization of the elements of the admissible control set. In other words, the important aspect of mixed strategies is that they are related to probability measures defined over the set of admissible controls. Thus part of our objective is to find for each player a best mixed strategy, where by mixed strategy is meant a cumulative distribution function, or c. d. f., defined over the set of admissible controls and parameterized as necessary by the state  $z$  and stage index  $i$ .

Since randomness was introduced via mixed strategies, the payoff function is a random variable and the state is a Markov sequence. Hence, it is reasonable that the contenders should wish to optimize a statistical function of the payoff  $J$ , in our case the mean. Therefore, in the games considered here, Player I is to use a strategy such that the minimum achievable mathematical expectation of  $J$  is maximized, and Player II will adopt a strategy which

minimizes the maximum achievable expectation of  $J$ . The mean of  $J$  for a given initial condition  $\underline{z}$  when both players use their best strategies is known (See, e. g., Blackwell and Girshick [6] and Fleming [53]) to satisfy the saddlepoint condition (2.6) for games of the type considered here and therefore is called the value  $w(\underline{z})$  of the game.

Let us make the above paragraphs more precise for the two variations of our basic game. To do this, we first introduce the notion of the truncated game  $i$ , which is the game which starts at stage  $i$  and position  $\underline{z}(i)$  and continues for  $N-i$  stages. The payoff for this game is

$$\begin{aligned}
 J_i &= J_i(\underline{z}; \underline{u}(i), \underline{u}(i+1), \dots, \underline{u}(N), \underline{v}(i), \underline{v}(i+1), \dots, \underline{v}(N)) \\
 &= g_{N+1}(\underline{z}(N+1)) + \sum_{k=i}^N g_k(\underline{z}(k), \underline{u}(k), \underline{v}(k))
 \end{aligned}
 \tag{3.4}$$

For the game of closed loop strategies, we seek optimal cumulative distribution functions (c. d. f.'s)  $F^0(\underline{u}(i) | \underline{z}(i), i)$  and  $G^0(\underline{v}(i) | \underline{z}(i), i)$  defined for the maximizer on  $U$  and for the minimizer on  $V$ , respectively, such that for each  $j=1, 2, \dots, N$ , and for each  $i=j, j+1, \dots, N$ , the value of the truncated game  $j$  is given by

(3.5)

$$w_j(\underline{z}(j)) = \int_V \int_U \dots \int_V \int_U J_j(\underline{z}(j); \underline{u}(j), \dots, \underline{u}(N), \underline{v}(j), \dots, \underline{v}(N))$$

$$dF^0(\underline{u}(N) | \underline{z}(N), N) dG^0(\underline{v}(N) | \underline{z}(N), N) \dots$$

$$\dots dF^0(\underline{u}(j) | \underline{z}(j), j) dG^0(\underline{v}(j) | \underline{z}(j), j)$$

$$= \min_{\substack{G_i \in \Gamma_i \\ i=j, \dots, N}} \int_V \int_U \dots \int_V \int_U J_j(\underline{z}(j); \underline{u}(j), \dots, \underline{u}(N), \underline{v}(j), \dots, \underline{v}(N))$$

$$dF^0(\underline{u}(N) | \underline{z}(N), N) dG_N^0(\underline{v}(N) | \underline{z}(N), N) \dots$$

$$\dots dF^0(\underline{u}(j) | \underline{z}(j), j) dG_j^0(\underline{v}(j) | \underline{z}(j), j)$$

$$= \max_{\substack{F_i \in \Phi_i \\ i=j, \dots, N}} \int_V \int_U \dots \int_V \int_U J_j(\underline{z}(j); \underline{u}(j), \dots, \underline{u}(N), \underline{v}(j), \dots, \underline{v}(N))$$

$$dF_N(\underline{u}(N) | \underline{z}(N), N) dG^0(\underline{v}(N) | \underline{z}(N), N) \dots$$

$$\dots dF_j(\underline{u}(j) | \underline{z}(j), j) dG^0(\underline{v}(j) | \underline{z}(j), j)$$

Here  $\Gamma_i$  and  $\Phi_i$  are the sets of all admissible conditional c. d. f.'s defined on V and U, respectively. That such a  $w_j(\underline{z})$  indeed exists is known from Fleming [53]; this function is discussed further in Chapter 5 when dynamic programming is considered.

For the game of open loop strategies, the players of the truncated game j must develop their strategies under the assumption that  $\underline{z}(i)$ ,  $i > j$ , may never be known to them. Hence, in this case the

c. d. f.'s sought can be conditioned only on  $\underline{z}(j)$  and on the player's own controls. We therefore need c. d. f.'s  $\hat{F}^0(\underline{u}(i)|\underline{z}(j), j; \underline{u}(j), \underline{u}(j+1), \dots, \underline{u}(i-1))$  and  $\hat{G}^0(\underline{v}(i)|\underline{z}(j), j; \underline{v}(j), \underline{v}(j+1), \dots, \underline{v}(i-1))$  for which the value function  $\hat{w}_j$  satisfies

$$\hat{w}_j(\underline{z}(j)) = \int \int \dots \int \int J_j(\underline{z}(j); \underline{u}(j), \dots, \underline{u}(N), \underline{v}(j), \dots, \underline{v}(N)) \quad (3.6)$$

$$\int \int \dots \int \int J_j(\underline{z}(j); \underline{u}(j), \dots, \underline{u}(N), \underline{v}(j), \dots, \underline{v}(N))$$

$$d\hat{F}^0(\underline{u}(N)|\underline{z}(j), j; \underline{u}(j), \dots, \underline{u}(N-1))$$

$$d\hat{G}^0(\underline{v}(N)|\underline{z}(j), j; \underline{v}(j), \dots, \underline{v}(N-1)) \dots$$

$$\dots d\hat{F}^0(\underline{u}(j)|\underline{z}(j), j) d\hat{G}^0(\underline{v}(j)|\underline{z}(j), j)$$

$$\min_{i=j, \dots, N} \hat{G}_i \epsilon \hat{F}_i \int \int \dots \int \int J_j(\underline{z}(j); \underline{u}(j), \dots, \underline{u}(N), \underline{v}(j), \dots, \underline{v}(N))$$

$$\int \int \dots \int \int J_j(\underline{z}(j); \underline{u}(j), \dots, \underline{u}(N), \underline{v}(j), \dots, \underline{v}(N))$$

$$d\hat{F}^0(\underline{u}(N)|\underline{z}(j), j; \underline{u}(j), \dots, \underline{u}(N-1))$$

$$d\hat{G}_N(\underline{v}(N)|\underline{z}(j), j; \underline{v}(j), \dots, \underline{v}(N-1)) \dots$$

$$\dots d\hat{F}^0(\underline{u}(j)|\underline{z}(j), j) d\hat{G}_j(\underline{v}(j)|\underline{z}(j), j)$$

$$\max_{i=j, \dots, N} \hat{F}_i \epsilon \hat{G}_i \int \int \dots \int \int J_j(\underline{z}(j); \underline{u}(j), \dots, \underline{u}(N), \underline{v}(j), \dots, \underline{v}(N))$$

$$\int \int \dots \int \int J_j(\underline{z}(j); \underline{u}(j), \dots, \underline{u}(N), \underline{v}(j), \dots, \underline{v}(N))$$

$$d\hat{F}_N(\underline{u}(N)|\underline{z}(j), j; \underline{u}(j), \dots, \underline{u}(N-1))$$

$$d\hat{G}^0(\underline{v}(N)|\underline{z}(j), j; \underline{v}(j), \dots, \underline{v}(N-1)) \dots$$

$$\dots d\hat{F}_j(\underline{u}(j)|\underline{z}(j), j) d\hat{G}^0(\underline{v}(j)|\underline{z}(j), j)$$

Again  $\hat{\Phi}_i$  and  $\hat{\Gamma}_i$  are the sets of admissible conditioned c. d. f.'s, but they are not identical to those of the closed-loop strategy case, which has a different structure. It is demonstrated in Chapter 5 that this case reduces to a parameterized static case, so that standard min-max theorems are applicable (e. g., see Blackwell and Girshtick [5]).

### 3.3 THE IMPORTANCE OF POLYNOMIAL N-STAGE GAMES

The class of games considered here is a special one: removal of the zero-sum assumption or introduction of stochastic observations or dynamics would create extremely difficult problems both of concept and of computation. Nevertheless, our games are not trivial. Two-person zero-sum games are good models of parlor games and satisfactory approximations of many other situations. Multistage dynamics are suitable for describing the manner in which many real situations effectively evolve. That the control vectors must be finite is eminently reasonable.

The polynomial approximation must be justified more subjectively. Polynomials are widely used in engineering work as the next step beyond simple linear models for many functions of interest can be approximated arbitrarily well by polynomials. Particularly when elaborate, aesthetically satisfying models prove insoluble, the solutions to polynomial models may be important for themselves and for the insight which they provide. It can be expected that solutions of polynomial games will have similar utility.

## CHAPTER 4

### THE SOLUTION OF SEPARABLE STATIC GAMES

In this chapter we consider the solution of games for which Player I selects a point  $\underline{u} \in U \subseteq E^m$ , Player II simultaneously selects  $\underline{v} \in V \subseteq E^n$ , and then Player II pays to Player I an amount defined by a function of the form

$$J(\underline{u}, \underline{v}) = \sum_{j=0}^{\nu} \sum_{i=0}^{\mu} a_{ij} r_i(\underline{u}) s_j(\underline{v}) \quad (4.1)$$

By making the coefficients  $a_{ij}$  functions of a state vector  $\underline{z}$ , we will in Chapter 5 relate this problem to the multistage game problem.

We remark that the game with payoff (4.1) is known to have a value and optimum strategies provided that  $J(\underline{u}, \underline{v})$  is continuous,  $U$  and  $V$  are closed and bounded, and mixed strategies defined on an infinite number of points are allowed. (See, for example, Blackwell and Girshick [6], Chapter 2). The results of this chapter will have the effect of proving this independently since they essentially demonstrate the value and strategies for the class of games considered.

#### 4.1 SEPARABLE PAYOFF FUNCTIONS AND THE MOMENT PROBLEM

Single-stage games with payoff functions defined by polynomials,

$$J(u, v) = \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} a_{ij} u^i v^j, \quad (4.2)$$

where  $u$  and  $v$  are scalars, are among the simplest examples of a

general class of games with separable payoffs, i. e., payoffs of the form

$$J(\underline{u}, \underline{v}) = \sum_{i=0}^M \sum_{j=0}^N a_{ij} r_i(\underline{u}) s_j(\underline{v}) \quad (4.1)$$

where  $r_i(\underline{u})$  and  $s_j(\underline{v})$  are continuous functions, and where  $\underline{u} \in U$ ,  $\underline{v} \in V$ , for  $U$  and  $V$  defined as unit hypercubes of dimension  $m$  and  $n$ , respectively.

$$\begin{aligned} U &= \{\underline{u} | u_i \in [0, 1], i=1, 2, \dots, m; \underline{u} \in E^m\} \\ V &= \{\underline{v} | v_i \in [0, 1], i=1, 2, \dots, n; \underline{v} \in E^n\} \end{aligned} \quad (4.3)$$

For general polynomial payoffs, in which our ultimate interest lies, the functions  $r_i(\underline{u})$  have the form

$$r_i(\underline{u}) = u_1^{k_{i1}} u_2^{k_{i2}} \dots u_m^{k_{im}}, \quad (4.4)$$

where the exponents  $k_{ij}$  are non-negative integers; the  $s_j(\underline{v})$  have analogous forms. The importance of separable payoffs is, as we shall develop below, the fact that the problem of determining optimal mixed strategies may be reduced to a problem of finding optimal vectors in certain convex sets.

To find solutions to the game with payoff (4.1), we will search among the classes of mixed strategies for the contestants, keeping in mind that pure strategies are special cases of mixed strategies. Thus let admissible strategies for Player I, the maximizer, consist of all cumulative distribution functions (c. d. f. 's) defined over the set  $U$ . This might also be pictured as the class of

joint distribution functions for the variables  $u_1, u_2, \dots, u_m$ . Let  $F(\underline{u})$  denote an admissible c. d. f. Similarly, let admissible strategies for Player II, the minimizer, consist of all c. d. f.'s defined on  $V$  and let  $G(\underline{v})$  be an element of this class. Then we may compute the expected value for  $J(\underline{u}, \underline{v})$  as

$$J(F, G) = \int_V \int_U J(\underline{u}, \underline{v}) dF(\underline{u}) dG(\underline{v}) \quad (4.5)$$

At this point we use the separability characteristic of  $J(\underline{u}, \underline{v})$  to rewrite (4.5) as

$$J(F, G) = \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} a_{ij} \int_V s_j(\underline{v}) dG(\underline{v}) \int_U r_i(\underline{u}) dF(\underline{u}) \quad (4.6)$$

If we define

$$r_i(F) = \int_U r_i(\underline{u}) dF(\underline{u}) \quad (4.7)$$

$$s_j(G) = \int_V s_j(\underline{v}) dG(\underline{v})$$

then (4.6) can be rewritten as

$$J(F, G) = \sum_{i=0}^{\mu} \sum_{j=0}^{\nu} a_{ij} r_i(F) s_j(G) \quad (4.8)$$

We may compress the notation somewhat by defining vectors

$$\underline{r}(F) = (r_0(F), r_1(F), \dots, r_{\mu}(F))^T \text{ and } \underline{s}(G) = (s_0(G), \dots, s_{\nu}(G))^T$$

and a matrix  $A = \{a_{ij}\}$   $i=0, 1, \dots, \mu$ ,  $j=0, 1, \dots, \nu$ , so that (4.8) becomes

$$J(F, G) = \underline{r}^T(F) A \underline{s}(G) \quad (4.9)$$

It is often convenient to remove the explicit dependence on the c. d. f. 's  $F(\underline{u})$  and  $G(\underline{v})$  by rewriting (4.9) as

$$J(\underline{r}, \underline{s}) = \underline{r}^T A \underline{s} \quad (4.10)$$

Let  $R$  denote the set of all vectors  $\underline{r}(F)$  obtained as  $F$  ranges over all admissible cumulative distribution functions on  $U$ , and let  $S$  similarly denote the set of all  $\underline{s}(G)$ . Since  $\underline{r}(F)$  and  $\underline{s}(G)$  are moments of their respective c. d. f. 's when the functions  $r_i(\underline{u})$  and  $s_j(\underline{v})$  are terms of polynomials, for the more general separable games it is useful to think of the functions as generalized moments and we shall often refer to them as such. By extension,  $R$  and  $S$  are called the generalized moment sets for Players I and II, respectively.

The importance of these transformations is that choosing a c. d. f. turns out to be equivalent to choosing generalized moments for a competitor. Thus our eventual problem, finding  $F^0(\underline{u})$  and  $G^0(\underline{v})$  such that

$$J(F, G^0) \leq J(F^0, G^0) \leq J(F^0, G) \quad (4.11)$$

where  $F$  and  $G$  are arbitrary admissible c. d. f. 's is equivalent to finding  $\underline{r}^0$  and  $\underline{s}^0$  such that

$$J(\underline{r}, \underline{s}^0) \leq J(\underline{r}^0, \underline{s}^0) \leq J(\underline{r}^0, \underline{s}) \quad (4.12)$$

for all  $\underline{r} \in R$  and  $\underline{s} \in S$ , and then finding distributions corresponding to the optimal  $\underline{r}^0$  and  $\underline{s}^0$ , provided, of course, that the saddlepoints (4.11) and (4.12) exist. This transformation of the problem is a key step on the path to solution of our separable games even though it is little more than a change of variable.

#### 4.2 ADMISSIBLE MOMENTS-THE SETS R AND S

The search for the saddlepoint implied by (4.12) requires that the sets  $R$  and  $S$  of admissible generalized moments be carefully characterized. They are by definition the sets of all moments generated by the classes of all cumulative probability distributions defined on the hypercubes  $U$  and  $V$ , respectively. The theorem of this section allows a simpler and more meaningful characterization of the sets, and is a generalization of a theorem of Dresher, Karlin, and Shapley [38]. We consider the set  $R$  and note that analogous results may be obtained for  $S$ .

The following well-known lemma is necessary for the proof of the theorem and is also used repeatedly in later sections. A proof is given by Karlin [40].

**Lemma A:** If  $[X]$  is the convex hull of an arbitrary set  $X$  in  $n$ -space, then every point of  $[X]$  may be represented as a convex combination of a most  $n+1$  points of  $X$ . Furthermore, if  $X$  is connected, then at most  $n$  points are needed.

In many applications of this lemma we are particularly interested in the fact that a finite convex representation of a point of the convex hull of a set is possible with the dimension of the representation being of secondary importance.

We return to our development of a characterization of the set  $R$  by defining the set  $C_R$  as the surface represented parametrically as a transformation via the functions  $r_i(\underline{u})$  of all points in  $U$ , that is,

$$C_R = \{ \underline{x} | \underline{x} \in E^{m+1}, \exists \underline{t} \in U \rightarrow \underline{x} = \underline{r}(\underline{t}) \} \quad (4.13)$$

With this set defined, we may proceed to the following theorem for which the proof is nearly identical to that for a less comprehensive theorem given by Karlin [40].

**Theorem 4.1.** The set  $R$  is the convex hull of the set  $C_R$  defined by equation (4.13).

**Proof:** Let  $D$  be the convex hull of  $C_R$ . Then we must prove that  $R = D$ .

(i) We prove first that  $R \subset D$ . Assume the contrary. Then there exists  $\underline{r}^0 \in R$  such that  $\underline{r}^0 \notin D$ . Now  $D$  is the convex hull of the continuous mapping of the closed convex set  $U$ , and therefore  $D$  is itself closed and convex.

But then there must be a hyperplane with normal vector  $\underline{h}$ , which strictly separates  $\underline{r}^0$  from  $D$ , i. e.,

$$\underline{h}^T \underline{r}^0 - \underline{h}^T \underline{r}(\underline{u}) \geq \delta > 0 \quad \text{for all } \underline{u} \in U \quad (4.14)$$

Since  $\underline{r}^0 \in R$ , there exists a c. d. f.  $F^0(\underline{u})$  such that

$$\int_U \underline{r}(\underline{u}) dF^0(\underline{u}) = \underline{r}^0 \quad (4.15)$$

If we average (4.14) using this distribution, we find

$$\begin{aligned} \underline{h}^T \underline{r}^0 \int_U dF^0(\underline{u}) - \underline{h}^T \int_U \underline{r}(\underline{u}) dF^0(\underline{u}) \\ = \underline{h}^T \underline{r}^0 - \underline{h}^T \underline{r}^0 \geq \delta \int_U dF^0(\underline{u}) = \delta > 0 \end{aligned} \quad (4.16)$$

which is clearly contradictory. Therefore,  $R \subset D$ .

(ii) To prove  $D \subset R$ , we choose an arbitrary  $\underline{r}^0 \in D$  and demonstrate a c. d. f. for which the generalized moments are  $\underline{r}^0$ . From Lemma A, since  $D$  is by definition the convex hull of  $C_R$ , we know that  $\underline{r}^0$  can be represented as a finite convex combination of points of  $C_R$ , each of which is an image of a point of  $U$ . Thus

$$\underline{r}^0 = \sum_{i=1}^{\mu+1} \alpha_i \underline{r}(\underline{u}_i), \quad \begin{cases} \alpha_i \geq 0 \\ \underline{u}_i \in U \\ \sum_{i=1}^{\mu+1} \alpha_i = 1 \end{cases} \quad i=1, \dots, \mu+1 \quad (4.17)$$

Now let  $I_{\underline{x}}(\underline{u})$  represent the degenerate c. d. f. such that, for  $\underline{x} \in U$ ,

$$dI_{\underline{x}}(\underline{u}) = 0 \quad \underline{u} \neq \underline{x} \quad (4.18)$$

$$\int_U dI_{\underline{x}}(\underline{u}) = 1$$

and define

$$F^0(\underline{u}) = \sum_{i=1}^{\mu+1} \alpha_i I_{\underline{u}_i}(\underline{u}), \quad (4.19)$$

where the  $\alpha_i$  and  $\underline{u}_i$  are those determined in (4.17).

Then it follows that

$$\int_U \underline{r}(\underline{u}) dF^0(\underline{u}) = \sum_{i=1}^{\mu+1} \alpha_i \underline{r}(\underline{u}_i) = \underline{r}^0. \quad (4.20)$$

Hence the c. d. f.  $F^0(\underline{u})$  yields  $\underline{r}^0$  and  $D \subset R$ . Combining this with the result of part (i), we have  $R = D$  as required.

An immediate corollary of this is the theorem of Dresher, et al [38], which was concerned as was the rest of their work, with scalar controls  $u$  and  $v$  for the competitors.

**Corollary**  
4.1-1:

When the control space  $U$  is one-dimensional, then  $R$  is the convex hull of the curve  $C_R$  whose parametric representation is  $\underline{r} = \{\underline{r}(t)\}$  for  $t \in [0, 1]$ .

Under some circumstances the general formulation of  $C_R$  given by (4.13) can be simplified. The set  $U$  can always be written as the cartesian product of smaller hypercubes. Suppose

$$U = U_1 \times U_2 \quad (4.21)$$

where  $U_1$  is  $m_1$ -dimensional,  $U_2$  is  $m_2$ -dimensional, and  $m_1 + m_2 = m$ , and assume that the functions  $r_i(\underline{u})$ ,  $i=0, 1, \dots, \mu$ , are such that if we write

$$\underline{u} = \begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \end{bmatrix}, \quad \underline{u}_1 \in U_1, \underline{u}_2 \in U_2, \quad (4.22)$$

then

$$r_i(\underline{u}) = r_i(\underline{u}_1) \quad i=0, 1, \dots, \mu_1 \quad (4.23)$$

$$r_i(\underline{u}) = r_i(\underline{u}_2) \quad i=\mu_1+1, \dots, \mu$$

Then if we define the surfaces

$$C_1 = \{ \underline{x} \mid \underline{x} \in E^{\mu_1+1}, x_i = r_i(\underline{t}) \text{ for some } \underline{t} \in U_1 \} \quad (4.24)$$

$$C_2 = \{ \underline{x} \mid \underline{x} \in E^{\mu-\mu_1}, x_i = r_{\mu_1+i}(\underline{t}) \text{ for some } \underline{t} \in U_2 \}$$

and let  $R_1, R_2$  be the sets of generalized moments corresponding to the first and second of (4.23), we have the following useful theorem:

**Theorem 4.2:** If there exists a decomposition of  $U$  such that

$$(4.23) \text{ holds, then } C_R = C_1 \times C_2 \text{ and } R = R_1 \times R_2$$

for  $C_1, C_2$  defined by (4.24).

**Proof:** The first statement follows directly from the definitions of  $C_R$ ,  $C_1$ , and  $C_2$ . The second statement is an immediate result of the fact that  $R$ ,  $R_1$ , and  $R_2$  are convex hulls of  $C_R$ ,  $C_1$ , and  $C_2$ , respectively, as is seen by using their definitions along with Theorem 4.1.

This simple theorem is particularly useful when the functions  $r_i(\underline{u})$  each depend upon only one component of  $\underline{u}$ , which we refer to as a situation with uncoupled controls and which is often useful as an approximation in engineering applications. Under these circumstances we may order the functions so that

$$\begin{aligned}
 r_i(\underline{u}) &= r_i(u_1) & i=0, 1, \dots, \mu_1 \\
 r_i(\underline{u}) &= r_i(u_2) & i=\mu_1+1, \dots, \mu_2 \\
 &\vdots & \\
 &\vdots & \\
 &\vdots & \\
 r_i(\underline{u}) &= r_i(u_m) & i=\mu_{m-1}+1, \dots, \mu_m
 \end{aligned}
 \tag{4.25}$$

Then by defining

$$\begin{aligned}
 C_1 &= \{ \underline{x} \mid \underline{x} \in E^{\mu_1+1}, x_i = r_i(t), t \in [0, 1] \} \\
 C_j &= \{ \underline{x} \mid \underline{x} \in E^{\mu_j - \mu_{j-1}}, x_i = r_{\mu_{j-1}+i}(t), t \in [0, 1] \} \\
 & \qquad \qquad \qquad j=2, 3, \dots, m
 \end{aligned}
 \tag{4.26}$$

and letting  $R_i$  be the convex hull of  $C_i$  for  $i=1, 2, \dots, m$ , we have the following corollary:

Corollary

4.2-1:

If the controls  $u_i$  are uncoupled, then the surface  $C_R$  is the cartesian product of the curves  $C_i$ , and  $R$  is the cartesian product of the convex hulls  $R_i$  of  $C_i$ ,  $i=1, 2, \dots, m$ , that is

$$C_R = C_1 \times C_2 \times C_3 \times \dots \times C_m \quad (4.27)$$

$$R = R_1 \times R_2 \times R_3 \times \dots \times R_m$$

**Proof:** The corollary follows from repeated application of Theorem 4.2.

Note that theorem 4.2 and its corollary are not trivially true; a general parameterized surface cannot always be represented as a product of parameterized subsurfaces.

#### 4.3 SOLUTIONS-THE METHOD OF CONVEX CONES

At this point we are ready to proceed with the development of solutions to our problem. We shall follow Dresher, et al, [38] for the early development and theorems 4.3 and 4.4. The key result of this section is theorem 4.5.

Let us briefly review our results so far. We have found that the problem of finding a saddlepoint in mixed strategies for  $J(\underline{u}, \underline{v})$  as given by (4.1) can be transformed to the problem of finding a saddlepoint in the generalized moments  $\underline{r}$  and  $\underline{s}$  for the function  $\underline{r}^T A \underline{s}$  where  $\underline{r} \in R$  and  $\underline{s} \in S$ . Furthermore, we have found that  $R$  is the convex hull of the set  $C_R$  defined parametrically by  $\underline{r}(\underline{u})$  as  $\underline{u}$  ranges over  $U$  and that  $S$  is the convex hull of an analogously defined

set  $C_S$ . The definitions of  $R$  and  $S$  imply that they are compact and convex.

Rather than augment the sets  $R$  and  $S$ , we shall make the convenient assumption that the functions  $r_i(\underline{u})$  and  $s_i(\underline{v})$  are such that

$$\begin{aligned} r_0(\underline{u}) &= 1 \\ s_0(\underline{v}) &= 1 \end{aligned} \tag{4.28}$$

so that if  $\underline{r} \in R$ , then  $r_0 = 1$  and if  $\underline{s} \in S$  then  $s_0 = 1$ . Also we define the sets  $\hat{S} \subset S$  and  $\hat{R} \subset R$  as the projected sets  $\hat{S} = \{\hat{\underline{s}} \mid \hat{\underline{s}} \in E^\nu, \hat{s}_i = s_i, i=1, 2, \dots, \nu \text{ for some } \underline{s} \in S\}$  and  $\hat{R} = \{\hat{\underline{r}} \mid \hat{\underline{r}} \in E^\mu, \hat{r}_i = r_i, i=1, 2, \dots, \mu \text{ for some } \underline{r} \in R\}$ . These notational conveniences are useful when considering convex cones and support hyperplanes and clearly lead to no loss of generality in our problem definition.

We begin the solution by defining the convex cones

$$\begin{aligned} P_R &= \{\underline{r} \mid \underline{r} \in E^{\mu+1}, \underline{r} = \lambda \underline{x} \text{ for some } \lambda \geq 0 \text{ and } \underline{x} \in R\} \\ P_S &= \{\underline{s} \mid \underline{s} \in E^{\nu+1}, \underline{s} = \lambda \underline{y} \text{ for some } \lambda \geq 0 \text{ and } \underline{y} \in S\} \end{aligned} \tag{4.29}$$

Geometrically, these are cones with vertices at the origin, and with cross-sections  $\hat{R}$  and  $\hat{S}$  at  $r_0=1, s_0=1$ , respectively. Associated with these cones are the dual cones defined by

$$\begin{aligned} P_R^* &= \{\underline{r} \mid \underline{r} \in E^{\mu+1}, \underline{r}^T \underline{x} \geq 0 \text{ for all } \underline{x} \in P_R\} \\ P_S^* &= \{\underline{s} \mid \underline{s} \in E^{\nu+1}, \underline{s}^T \underline{y} \geq 0 \text{ for all } \underline{y} \in P_S\} \end{aligned} \tag{4.30}$$

Note that  $P_R^*$  is a closed convex cone, and that  $\underline{r} \in P_R^*$  is a boundary point of  $P_R^*$  only if there exists  $\underline{x} \in R$  such that  $\underline{r}^T \underline{x} = 0$ . Analogous statements hold for  $P_S^*$ .

The relationships of the cones and dual cones are worth amplifying. Since  $P_R$  is a convex cone with vertex at the origin, if  $\underline{r}^0$  is an element of its boundary, then there will exist a hyperplane of support  $H$  to  $P_R$  at  $\underline{r}^0$  which contains the origin. Hence,  $H = \{\underline{x} | \underline{h}^0^T \underline{x} = 0, \underline{x} \in E^{n+1}\}$  for an appropriate  $\underline{h}^0$  such that

$$\begin{aligned} \underline{h}^0^T \underline{r}^0 &= 0 \\ \underline{r}^0^T \underline{r} &\geq 0, \underline{r} \in P_R \end{aligned} \tag{4.31}$$

The representation  $\underline{h}^0$  of  $H$  thus belongs to  $P_R^*$ , and in fact it can be shown to be a boundary point of  $P_R^*$ . Equations (4.31) also hold if  $\underline{r}^0 \in R$  and  $\underline{r} \in R$ , provided that only support hyperplanes  $H$  to  $R$  which pass through the origin are considered. In fact, a little reflection reveals that  $H$  can be generated in  $E^n$  by using support hyperplanes to  $\hat{R}$  which are not constrained to pass through the origin, a fact which follows from the definition of  $\hat{R}$ . Therefore, support hyperplanes to  $\hat{R}$  are closely related to the support hyperplanes of  $R$  and of  $P_R$ , a useful property which is exploited in later sections. Furthermore, since  $(P_R^*)^* = P_R$ , as is easily shown, the support hyperplanes of  $P_R^*$  correspond to boundary points of  $P_R$  and, ultimately, of  $R$  and of  $\hat{R}$ . The situation for  $S$  and  $P_S^*$  is, of course, analogous.

Assume that it is known that the value of the game under consideration is zero, that is

$$\min_{\underline{s} \in S} \max_{\underline{r} \in R} \underline{r}^T A \underline{s} = 0 \quad (4.32)$$

Define the set

$$S(A, R) = \{ \underline{s} \mid \underline{s} \in E^{N+1}, \underline{s} = A^T \underline{r} \text{ for some } \underline{r} \in R \}, \quad (4.33)$$

which is the image under the linear transformation represented by the matrix  $A^T$  of the set  $R$ .

The following two theorems were originally due to Dresher, et al, [38] and are fundamental to our theory. Brief proofs are given because they help illustrate the interrelationships of the sets. The proofs are basically due to Karlin [40].

**Theorem 4.3:** For the game of value zero, if  $R^0$  denotes the set of optimal strategies for the maximizing player, then

$$S(A, R^0) = S(A, R) \cap P_S^* \quad (4.34)$$

Furthermore,  $S(A, R)$  does not overlap  $P_S^*$  in its interior.

**Proof:** Assume to the contrary that the two sets overlap. Then there exists  $\underline{r}^0 \in R$  such that  $\underline{r}^0{}^T A \underline{s} \geq \delta > 0$  for all  $\underline{s} \in S$ , implying that the game has a value of at least  $\delta$ , a contradiction. Thus the second

statement is established.

Since optimal strategies exist,  $R^0$  is not empty. Let  $\underline{r}^0 \in R^0 \subset R$ , and note that optimality implies  $\underline{r}^{0T} A \underline{s} \geq 0$  for all  $\underline{s} \in S$ , so that  $A^T \underline{r}^0 \in P_S^*$ . Thus  $S(A, R^0) \subset S(A, R) \cap P_S^*$ .

Conversely,  $A^T \underline{r} \in P_S^*$  for some  $\underline{r} \in R$  implies  $\underline{r}^0T A \underline{s} \geq 0$  for all  $\underline{s} \in S$ , which gives  $\underline{r}^0 \in R^0$ . Therefore,  $S(A, R^0) \supset S(A, R) \cap P_S^*$ , and the proof is complete.

**Theorem 4.4:** The separating planes of  $S(A, R)$  and  $P_S^*$  are in one-to-one correspondence with the optimal strategies for the minimizing player.

**Proof:** Let  $S^0$  be the set of optimal strategies for the minimizer. For any  $\underline{s}^0 \in S^0$ , we have  $\underline{r}^T A \underline{s}^0 \leq 0$  for all  $\underline{r} \in R$  and  $\underline{h}^T \underline{s}^0 \geq 0$  for all  $\underline{h} \in P_S^*$ . Thus  $\underline{s}^0$  represents a separating hyperplane.

Conversely, since  $S(A, R)$  and  $P_S^*$  are in contact, any separating hyperplane must be a support hyperplane to both. Let  $\underline{s}^*$  represent such a hyperplane. Then  $\underline{r}^T A \underline{s}^* \leq 0$  for all  $\underline{r} \in R$ , and  $\underline{h}^T \underline{s}^* \geq 0$  for all  $\underline{h} \in P_S^*$ . The latter fact implies  $\underline{s}^* \in P_S$  so that by suitable scaling we may take  $\underline{s}^* \in S$ . But this together with  $\underline{r}^T A \underline{s}^* \leq 0$  gives that  $\underline{s}^* \in S^0$ , and the proof is finished.

In general, of course, a game will have a non-zero value

$$w = \min_{\underline{s} \in S} \max_{\underline{r} \in R} \underline{r}^T A \underline{s} \quad (4.35)$$

Define a vector  $\underline{\alpha} \in E^{v+1}$  such that  $\alpha_0 = \alpha$ , and  $\alpha_i = 0$ ,  $i=1, 2, \dots, v$ .

Modify the set (4.33) by defining a new set

$$S(A, R, \alpha) = \{ \underline{s} \in E^{v+1}, \underline{s} = A^T \underline{r} - \underline{\alpha} \text{ for some } \underline{r} \in R \quad (4.36)$$

$$\text{and } \alpha_0 = \alpha, \alpha_i = 0, i=1, 2, \dots, v \}$$

The following theorem is fundamental for our solution methods.

**Theorem 4.5:** For the game  $\underline{r}^T A \underline{s}$ ,  $\underline{r} \in R$  and  $\underline{s} \in S$ , the value  $w$  is determined by

$$w = \max \{ \alpha \mid P \neq \emptyset \cap S(A, R, \alpha) \neq \emptyset \} \quad (4.37)$$

where  $\emptyset$  is the empty set.

**Proof:**

We note that the parameter  $\alpha$  has the effect of translating the set  $S(A, R)$  parallel to the  $s_0$ -axis. Because  $r_0 = 1$  for  $\underline{r} \in R$ , this same effect may be had by modifying the  $a_{00}$  element of the matrix  $A$ . Let us do so, creating the matrix  $A_\alpha$

$$A_\alpha = \{ a_{ij} - \alpha_{ij} \} \quad \begin{matrix} a_{ij} \in A \\ \alpha_{ij} = \begin{cases} \alpha & i=j=0 \\ 0 & \text{Otherwise} \end{cases} \end{matrix} \quad (4.38)$$

so that

$$S(A, R, \alpha) = S(A_\alpha, R, 0) = S(A_\alpha, R) \quad (4.39)$$

If we consider the game defined by  $A_\alpha$ ,  $R$ , and  $S$ , we find, since  $s_j = 1$  for  $\underline{s} \in S$ , that

$$\min_{\underline{s} \in S} \max_{\underline{r} \in R} \underline{r}^T A_\alpha \underline{s} = \min_{\underline{s} \in S} \max_{\underline{r} \in R} \underline{r}^T A \underline{s} - \alpha = w - \alpha \quad (4.40)$$

From this equation, our proof follows readily. If  $\alpha > w$ , then the value of the game with matrix  $A_\alpha$  is negative, implying that there exists  $\underline{s}^0 \in S$  such that  $\underline{r}^T A_\alpha \underline{s}^0 < 0$  for all  $\underline{r} \in R$ . Since  $\underline{h} \in P_S^*$  means  $\underline{h}^T \underline{s} \geq 0$ , it must be that  $A_\alpha^T \underline{r} \notin P_S^*$  for all  $\underline{r} \in R$ , or equivalently that  $P_S^* \cap S(A, R, \alpha) = \emptyset$ .

On the other hand,  $\alpha \leq w$  implies that the game (4.40) has a non-negative value. Thus there will exist  $\underline{r}^0 \in R$  such that  $\underline{r}^0^T A_\alpha \underline{s} \geq 0$  for all  $\underline{s} \in S$ . This implies  $A_\alpha^T \underline{r}^0 \in P_S^*$ , so that  $P_S^* \cap S(A, R, \alpha) \neq \emptyset$ . Therefore,  $w$  is the largest value of  $\alpha$  such that the intersection is non-empty.

From (4.40) we see that as a result of our notation the game with matrix  $A_w$  has value zero. Theorems (4.3) and (4.4) can be used to determine the optimum strategy sets  $R^0$  and  $S^0$  for this game, and since  $w$  is a simple translation of the set  $S(A_w, R)$ , for the original game with matrix  $A$ . The three theorems form, therefore, the foundation of a solution technique: translate

$S(A, R)$  until it shares only boundary points with  $P_S^*$ . Then the points of intersection determine  $R^0$ , the amount of translation is the value of the game, and the separating hyperplanes define  $S^0$ .

#### 4.4 GEOMETRIC AND ALGEBRAIC CONSIDERATIONS FOR SIMPLE MOMENT SPACES

We have now established the essence of a solution technique for the problem of finding a saddlepoint in mixed strategies of the mean of the payoff  $J(\underline{u}, \underline{v})$  in equation (4.1). In the remainder of this chapter are discussed some of the important considerations in applying the method, including algebraic and geometric descriptions of some of the sets, numerical approximations to solutions, and actual generation of the required probability distribution functions. Of necessity many of the results concern special cases for, as we shall see, characterization of the general problem is often difficult.

In this section we develop more detailed descriptions of the sets  $K$  and  $P_R^*$ . As usual, analogous results hold for  $S$  and  $P_S^*$ . Although we consider mostly special polynomial cases and, in fact, show the difficulty of applying our methods to more general problems, we must remember that Theorem 4.1 is true in general and can always be applied to generate  $R$  and that  $P_R^*$  can be developed directly from its definition, equation (4.30). We continue to assume that  $r_0 = 1$ .

Let us first consider the set  $R$  under the condition that  $u$  is one-dimensional and

$$r_i(u) = u^i \quad i=0, 1, \dots, \mu \quad (4.41)$$

This corresponds to a scalar control for the maximizer, and was considered by Karlin and Shapley [41], whose development we follow.

For convenience define vectors  $\underline{t}_j$

$$\underline{t}_j = (1, t_j, t_j^2, t_j^3, \dots, t_j^\mu)^T, \quad t_j \in [0, 1] \quad (4.42)$$

and note that  $C_R$  is the set of all such vectors. Assume  $\hat{\underline{r}}^0$  belongs to the boundary of  $\hat{R}$ , and let  $\underline{h}^0$  represent a support hyperplane to  $\hat{R}$  at  $\hat{\underline{r}}^0$ . Then

$$\begin{aligned} \underline{h}^{0T} \underline{r}^0 &= 0 & \underline{r}^0 &= \begin{bmatrix} 1 \\ \hat{\underline{r}}^0 \end{bmatrix} \\ \underline{h}^{0T} \underline{r} &\geq 0 & \text{for all } \underline{r} \in R \end{aligned} \quad (4.43)$$

will hold for this  $\underline{h}^0$ . But by Lemma A,

$$\underline{r}^0 = \sum_{i=1}^{\mu+1} \alpha_i \underline{t}_i \quad (4.44)$$

for suitable  $\underline{t}_i \in C_R$ , where  $\sum_{i=1}^{\mu+1} \alpha_i = 1$  and  $\alpha_i \geq 0$ ,  $i=1, 2, \dots, \mu+1$ .

Substituting (4.44) into (4.43)

$$\sum_{i=1}^{\mu+1} \alpha_i \underline{h}^{0T} \underline{t}_i = 0 \quad (4.45)$$

which gives, for all  $i$  such that  $\alpha_i > 0$ ,

$$\underline{h}^{0T} \underline{t}_i = 0 \quad (4.46)$$

since  $\underline{t}_j \in C_R \subset R$  implies  $\underline{h}^{\circ T} \underline{t}_j \geq 0$  for all  $j$ . Therefore, we may state that all points  $\underline{t}_i$  which appear nontrivially ( $\alpha_i > 0$ ) in the representation of  $\underline{r}^{\circ}$  also lie in the hyperplane represented by  $\underline{h}^{\circ}$ . Furthermore, all points  $\underline{r}$  which belong to the boundary of  $R$  and which are convex combinations of points  $\underline{t}_i$ ,  $i=1, 2, \dots, k$ , for some  $k \leq \mu+1$  lie in the hyperplane defined by

$$\underline{h}^{\circ T} \underline{t}_j = 0 \quad j=1, 2, \dots, k \quad (4.47)$$

With the above basic facts established, we proceed to develop a representation for  $\underline{h}^{\circ}$ . The requirement on  $\underline{h}^{\circ}$  represented by (4.43) implies that

$$\underline{h}^{\circ T} \underline{t} \geq 0 \quad (4.48)$$

for all  $t \in [0, 1]$ . This is a polynomial in  $t$  by definition of  $\underline{t}$ , and the inequality implies that any root of the polynomial on the open interval  $(0, 1)$  must be a double root. Thus there can be at most  $[\frac{\mu}{2}]$  zeros of (4.48) in  $(0, 1)$ , where  $[x]$  is the largest integer less than or equal to  $x$ . The roots corresponding to  $t=0$  and  $t=1$ , if any, may be single roots.

We notice that we may confine our attention to hyperplanes for which (4.48) has exactly  $\mu$  zeros in  $[0, 1]$ . This follows from the observation that, for example, a boundary point  $\underline{r}$  with representation in terms of points  $\underline{t}_i$ ,  $i=1, 2, \dots, k < [\frac{\mu}{2}]$  can be represented in terms of points  $\underline{t}_i$ ,  $i=1, 2, \dots, [\frac{\mu}{2}]$  when the additional points are given weightings  $\alpha_i=0$ ,  $i=k+1, \dots, [\frac{\mu}{2}]$ . This is

equivalent to selecting a particular support hyperplane when there is not a unique support hyperplane. Thus we come to two cases, depending upon whether  $\mu$  is odd or even.

Case 1:  $\mu$  even. The hyperplanes of interest will have either (a)  $\frac{\mu}{2}$  distinct roots in  $(0, 1)$  or will have (b)  $\frac{\mu}{2} - 1$  distinct roots in  $(0, 1)$  plus single roots of  $t=0$  and  $t=1$ . Therefore, the hyperplane will have elements implied by

$$(a) \quad \underline{h}^T \underline{t} = \alpha \prod_{j=1}^{\frac{\mu}{2}} (t - t_j)^2 \quad \alpha > 0$$

(4. 49)

$$(b) \quad \underline{h}^T \underline{t} = \alpha t(1 - t) \prod_{j=1}^{\frac{\mu}{2} - 1} (t - t_j)^2 \quad \alpha > 0$$

which result from simply writing out the polynomials in different form.

Case 2:  $\mu$  odd. The hyperplanes of interest have  $\frac{\mu-1}{2}$  distinct roots of (4. 48) in  $(0, 1)$  plus either (a) a single root at  $t=0$  or (b) a single root at  $t=1$ . The elements of  $\underline{h}$  will be implied by

$$(a) \quad \underline{h}^T \underline{t} = \alpha t \prod_{j=1}^{\frac{\mu-1}{2}} (t - t_j)^2 \quad \alpha > 0$$

(4. 50)

$$(b) \quad \underline{h}^T \underline{t} = \alpha(1 - t) \prod_{j=1}^{\frac{\mu-1}{2}} (t - t_j)^2 \quad \alpha > 0$$

In either Case 1 or Case 2, the elements of  $\underline{h}$  may be found in terms of the roots  $\underline{t}_j$  by simply matching coefficients. Therefore,  $\underline{h}$  may be found explicitly in terms of a set of parameters. For a given  $\mu$ , then we may find all support hyperplanes to  $\hat{R}$  by considering both type (a) and type (b) hyperplanes and allowing the roots  $\underline{t}_j$  to vary over  $(0, 1)$ . We shall find occasion to refer to the type (a) and (b) hyperplanes as lower and upper support hyperplanes, respectively. As a memory aid, we note that upper supports always have a single root at  $t=1$ .

To clarify the ideas developed so far, we present a simple example. Suppose  $\mu=2$ , so that  $C_R = \{\underline{t} | t_0 = 1, t_1 = t, t_2 = t^2; t \in [0, 1]\}$  and  $R$  is the convex hull of  $C_R$ . Then for any  $\underline{h}$ , either

$$\underline{h}^T \underline{t} = \alpha(t - t_1)^2 \quad t_1 \in (0, 1)$$

or

$$\underline{h}^T \underline{t} = \alpha t(1 - t)$$

These equations imply lower support planes of the form

$$\underline{h} = \alpha \begin{bmatrix} t_1^2 \\ -2t_1 \\ 1 \end{bmatrix} \quad t_1 \in (0, 1), \alpha > 0$$

and upper planes of the form

$$\underline{h} = \alpha \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \alpha > 0$$

We may now use our knowledge of the support hyperplanes to characterize  $\hat{R}$  in two ways. First, since  $\hat{R}$  is convex, it is determined by the intersection of the half-space defined by its support hyperplanes. Thus we may determine if a candidate point  $\underline{r}$  belongs to  $R$  by checking whether

$$\begin{aligned} \underline{h}_a^T(\mu; 0; t_1, t_2, \dots, t_{\lfloor \frac{\mu}{2} \rfloor}) \underline{r} &\geq 0 \quad \text{all } t_i \in (0, 1) \\ \underline{h}_b^T(\mu; 0; 1; t_1, t_2, \dots, t_{k(\mu)}) \underline{r} &\geq 0 \quad \text{all } t_i \in (0, 1) \end{aligned} \quad (4.51)$$

where  $\underline{h}_a$  and  $\underline{h}_b$  are the explicit representations of the relevant lower and upper support planes in terms of the parameters  $t_i$ , and  $k(\mu) = \lfloor \frac{\mu}{2} \rfloor$  for  $\mu$  odd and  $k(\mu) = \frac{\mu}{2} - 1$  for  $\mu$  even. This interpretation is exploited in the next section.

Second, and perhaps more important, the development of the representation of  $\underline{h}$  suggests what the boundary of  $R$  looks like. Examination of the arguments indicates that  $R$  will have a lower surface consisting of all convex combinations of all sets of exactly  $\lfloor \frac{\mu}{2} \rfloor$  points  $\underline{t}$ ,  $t \in (0, 1)$  and, if  $\mu$  is odd, the point  $\underline{t}$  for  $t=0$ . Also,  $R$  will have an upper surface consisting of all convex combinations of the point  $\underline{t}=1$ ,  $k(\mu)$  points generated by  $t$  in  $(0, 1)$ , and, if  $\mu$  is even, the point generated by  $t=0$ . Thus if  $\mu=2$ ,  $R$  has lower boundary defined by points  $\underline{t}$ ,  $t \in (0, 1)$ , and upper boundary defined by all points on the line segment from  $(1 \ 0 \ 0)^T$  to  $(1 \ 1 \ 1)^T$ . If  $\mu=3$ ,  $R$  has lower boundary defined by all points on the line segments from  $(1 \ 0 \ 0 \ 0)^T$  to  $(1 \ t \ t^2 \ t^3)^T$  and upper boundary defined by line segments from  $(1 \ 1 \ 1 \ 1)^T$  to  $(1 \ t \ t^2 \ t^3)^T$ .

The above discussion is easily extended to the case of uncoupled controls, equation (4.25), since by the use of Corollary 4.2-1 it is known that  $R$  is a cartesian product of sets  $R_i$ . Thus if each function  $r_i$  has the form

$$r_i(\underline{u}) = u_j^{k_{ij}} \quad i=1, 2, \dots, \mu \quad (4.52)$$

for some admissible integers  $j$  and  $k_{ij} > 0$ , and if we then order these functions in increasing  $j$  and for each  $j$  order the functions in increasing  $k_{ij}$ , then each  $R_j$  will, except for the constant term implied by  $r_0=1$ , be like the set  $R$  for the scalar control considered above. Explicitly we define

$$\hat{R}_j = \{\underline{x} | \underline{x} \in E^{\mu_j}, x_i = t^i, i=1, 2, \dots, \mu_j, t \in [0, 1]\} \quad (4.53)$$

so that we have  $R = \{1\} \times \hat{R}_1 \times \hat{R}_2 \times \dots \times \hat{R}_m$  and, by implication,  $\sum_{j=1}^m \mu_j = \mu$ . (This latter assumption is made without loss of generality, since the payoff may be augmented to make it true.)

Then it is easy to show that  $\underline{h} \in E^{\mu+1}$  such that  $\hat{h}$  supports  $\hat{R}$ ,

$$\underline{h}^T \underline{r} \geq 0 \quad \text{all } \underline{r} \in R$$

$$\underline{h}^T \underline{r}^0 = 0 \quad \text{some } \underline{r}^0 \in \hat{R}$$

implies, for  $j=1, 2, \dots, m$  and proper choice of  $h_{0j}$ ,

$$h_{0j} + \underline{h}_j^T \underline{r}_j \geq 0 \quad \text{all } \underline{r}_j \in R_j$$

$$h_{0j} + \underline{h}_j^T \underline{r}_j^0 = 0 \quad \underline{r}_j^0 \in \hat{R}_j$$

where

$$\underline{r}^o = \begin{bmatrix} 1 \\ \underline{r}_1^o \\ \underline{r}_2^o \\ \cdot \\ \cdot \\ \underline{r}_m^o \end{bmatrix}, \quad \sum_{j=1}^m h_{0j} = h_0$$

Hence, the hyperplane must support each of the sets  $\hat{R}_j$  individually. Thus the character of each of the sets  $R_j$  is established, as is the character and potential parameterization of the support hyperplanes.

Of particular interest is the fact that each  $\hat{R}_j$  has an upper and a lower surface, and therefore we may think of  $R$  as having  $2^m$  surfaces and of there being  $2^m$  types of hyperplanes supporting  $R$ . Each surface and each hyperplane type can be explicitly generated by choosing an upper or lower surface and the corresponding hyperplane set for each  $\hat{R}_j$ ,  $j=1, 2, \dots, m$ , for each combination of "upper" and "lower."

The construction of  $R$  when the controls are coupled does not appear to be amenable to analysis of the type used above. A pair of simple examples will help illustrate the difficulties. For example, let

$$\underline{r}(\underline{u}) = \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_1 u_2 \end{bmatrix} \quad u_i \in [0, 1], \quad i=1, 2 \quad (4.54)$$

Then, as sketched in Figure 4-1 for the cross-section  $r_0 = 1$ , we find that  $R$  is the polygon with vertices

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (4.55)$$

where  $C_R$  is the surface given parametrically by

$$\begin{bmatrix} 1 \\ t_1 \\ t_2 \\ t_1 t_2 \end{bmatrix} \quad t_i \in [0, 1], \quad i=1, 2 \quad (4.56)$$

For example 2, let

$$\underline{r}(\underline{u}) = \begin{bmatrix} 1 \\ u_1^2 \\ u_2^2 \\ u_1 u_2 \end{bmatrix} \quad u_i \in [0, 1], \quad i=1, 2 \quad (4.57)$$

Then, as sketched in Figure 4-2,  $C_R$  is given parametrically by

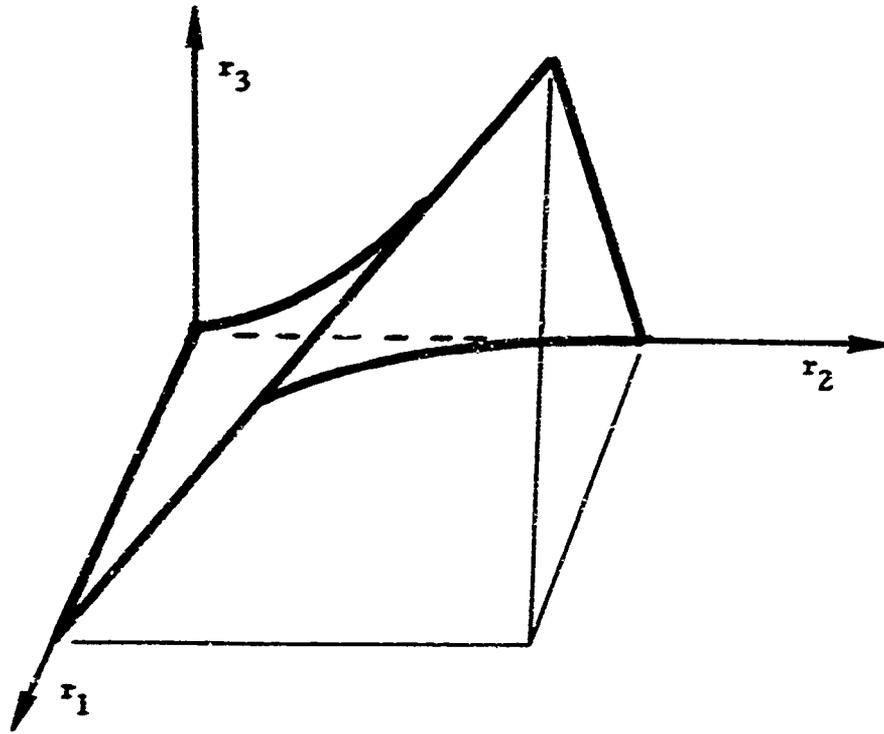
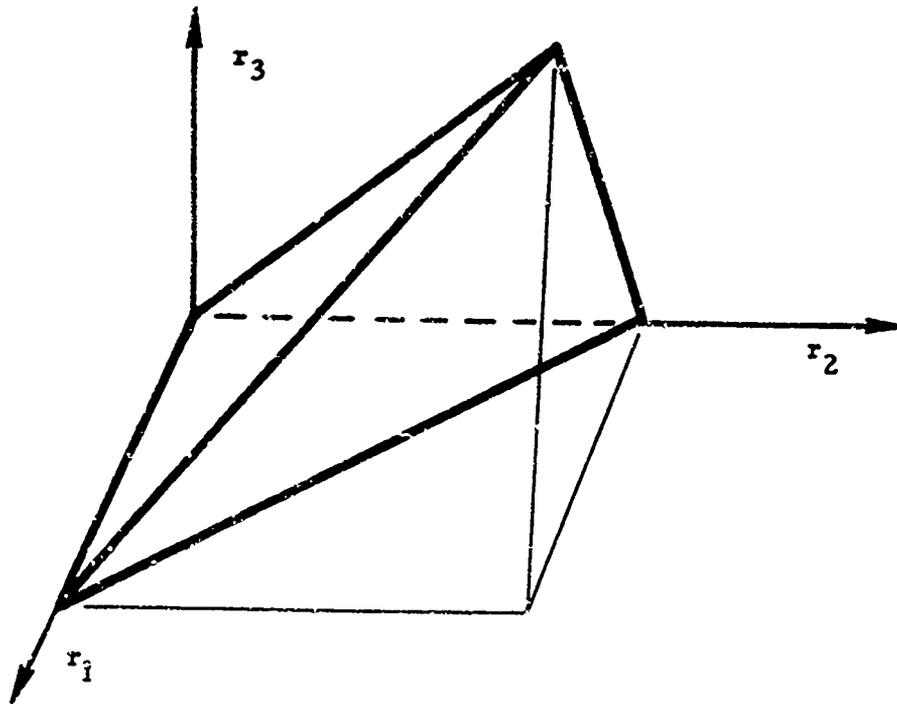


Figure 4.1. The sets  $\hat{C}_R$  (above) and  $\hat{R}$  (below) for Example i.



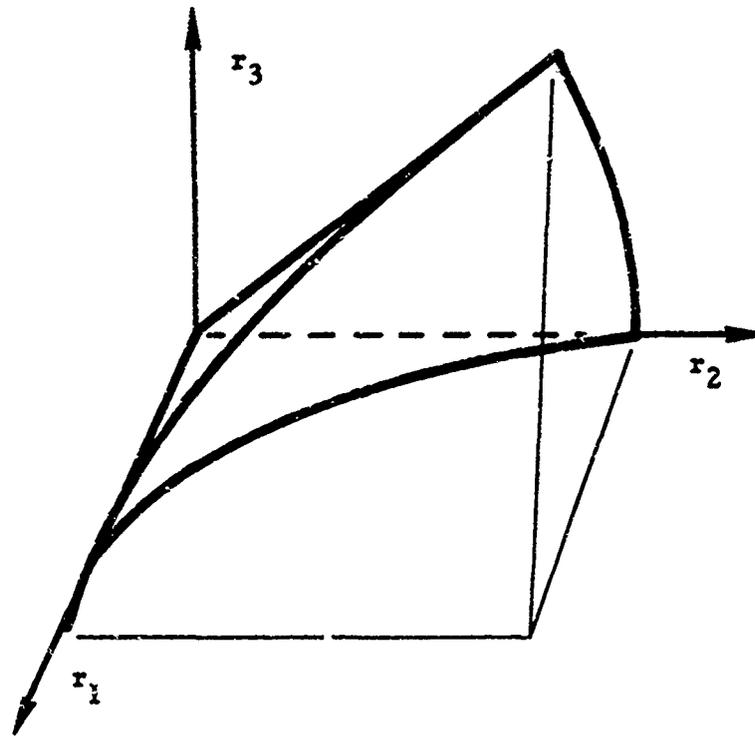
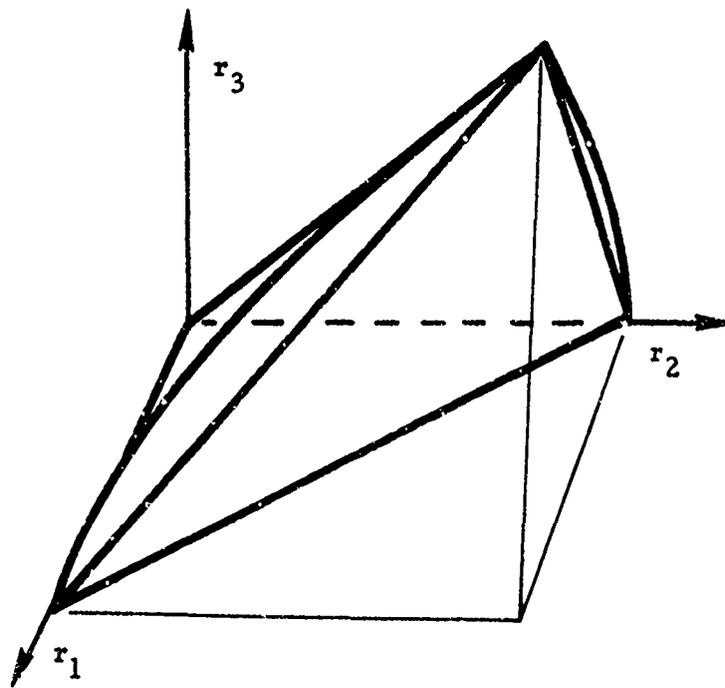


Figure 4-2. The sets  $\hat{C}_R$  (above) and  $\hat{R}$  (below) for example 2.



$$\begin{bmatrix} 1 \\ t_1^2 \\ t_2^2 \\ t_1 t_2 \end{bmatrix}$$

(4.58)

and the surfaces of  $R$  are (a) the surface  $C_R$ , and portions of the planes (b)  $r_3 = 0$ , (c)  $r_1 = 1$ , (d)  $r_2 = 1$ , (e)  $r_1 + r_2 - r_3 = 1$ .

In comparing examples 1 and 2, we see first that  $C_R$  is not necessarily a boundary surface of  $R$ , although it may be. Furthermore, the examples do not even have the same number of sets of support planes, since Example 1 has four sets and Example 2 has five sets.

Because of the apparent lack of common properties in the two examples, it appears likely that construction of  $R$  must usually be done on a case by case basis using Theorem 4.1. Naturally, other important special cases besides those of scalar and uncoupled controls which we have discussed may be characterizable.

At this point we turn from the set  $R$  to the dual cone  $P_R^*$ . Since it is the boundary of the dual cone which is of importance for problem solutions (Theorem 4.3), we shall be particularly concerned with generating that boundary. We establish the following theorem as being particularly useful in this regard.

Theorem 4.6: The dual cone  $P_R^*$  may be generated using the surface  $C_R$ , that is,

$$P_R^* = \{ \underline{x} | \underline{x}^T \underline{y} \geq 0 \text{ for all } \underline{y} \in C_R \} \quad (4.59)$$

Proof:

Let  $R$  be the convex hull of  $C_R$  and let  $P_R^*$  be the dual cone corresponding to the cone  $P_R$  generated by  $R$ . Let  $P_C^*$  denote the set defined by the right hand side of (4.59). Then we must prove that  $P_R^* = P_C^*$ . Since  $C_R \subset P_R$ , it is clear that the definition of  $P_C^*$  is less restrictive than that of  $P_R^*$  so that  $P_R^* \subset P_C^*$ .

Conversely, let  $\underline{h} \in P_C^*$ . By Lemma A any point  $\underline{r}^0 \in R$  may be represented by a finite convex combination of points  $\underline{r}_i$  of  $C_R$ , i. e.,

$$\underline{r}^0 = \sum_{i=1}^k \alpha_i \underline{r}_i \quad \sum_{i=1}^k \alpha_i = 1 \quad \alpha_i > 0$$

for some integer  $k \leq \mu+1$ . Furthermore, any point  $\underline{x} \in P_R$  may be represented as a non-negative scalar multiple of some point  $\underline{r}^0 \in R$ ,  $\underline{x} = \lambda \underline{r}^0$ . Thus for arbitrary  $\underline{x} \in P_R$  we have for  $\underline{h} \in P_C^*$ ,

$$\underline{h}^T \underline{x} = \lambda \underline{h}^T \underline{r}^0 = \lambda \sum_{i=1}^k \alpha_i \underline{h}^T \underline{r}_i \quad (4.60)$$

Since  $\lambda$  and  $\alpha_i$  are non-negative, and  $\underline{h}^T \underline{r}_i \geq 0$  because  $\underline{h} \in P_C^*$  and  $\underline{r}_i \in C_R$  by definition, equation (4.60) is non-negative. Therefore  $P_C^* \subset P_R^*$  and our proof is complete.

Use of this theorem leads to a general technique for generating  $P_R^*$ , one that will be used for the analogous set  $P_S^*$  in the next section. For each point  $\underline{r} \in C_R$ , we may generate a half-space

$$H(\underline{r}) = \{ \underline{x} \mid \underline{x} \in E^{\mu+1}, \underline{x}^T \underline{r} \geq 0 \} \quad (4.61)$$

The intersection of all such half-spaces constitutes the set  $P_R^*$ . The boundary of  $P_R^*$  can consist only of points  $\underline{x}$  for which  $\underline{x}^T \underline{r} = 0$  for at least one  $\underline{r} \in C_R$ , although the existence of such an  $\underline{r}$  does not guarantee that  $\underline{x}$  is a boundary point. The generation of  $P_R^*$  by this approach can obviously be tedious.

For the special case of polynomials and scalar controls, we are able to say slightly more about  $P_R^*$ . In this case, we are working with polynomials

$$\underline{h}^T \underline{t} \geq 0 \quad (4.62)$$

where  $\underline{t} = (1 \ t \ t^2 \ \dots \ t^\mu)^T$ , since  $C_R$  is defined by the vectors  $\underline{t}$ , and where  $\underline{h} \in P_R^*$ . To be on the boundary of  $P_R^*$ , a vector  $\underline{h}$  must have a corresponding  $\underline{t}_h$  such that

$$\underline{h}^T \underline{t}_h = 0 \quad (4.63)$$

However, since (4.62) must hold for all  $\underline{t}$  for a given  $\underline{h}$ , it follows that if  $t_h \in (0, 1)$ ,

$$(a) \quad \frac{d}{dt} \underline{h}^T \underline{t} \Big|_{\underline{t}=\underline{t}_h} = 0 \quad (4.64)$$

$$(b) \quad \frac{d^2}{dt^2} \underline{h}^T \underline{t} \Big|_{t=t_h} \geq 0 \quad (4.64)$$

As we shall see in later sections, the relationships (4.63) and (4.64a) can be used to find  $\underline{h}$  in terms of  $t \in (0, 1)$  for some regions of  $P_R^*$ . The usual extensions to include end points  $t = 0$  and  $t = 1$ , and to consider uncoupled controls using cartesian products may be made.

We remark that since points of the boundary of  $P_R^*$  correspond to support hyperplanes, the discussion at the beginning of this section concerning support hyperplanes for  $R$  can in principle be used to find  $P_R^*$ . However, considerable additional work is needed because that discussion did not use all support hyperplanes when a choice was possible. The unused planes were unneeded for defining  $R$ , but are essential for defining  $P_R^*$ . Therefore the method outlined here appears preferable operationally. Theorems related to extending the hyperplane approach for scalar controls may be found in Shapley and Karlin [41].

#### 4.5 NUMERICAL SOLUTIONS AND AN APPROXIMATION TECHNIQUE

Actual solution of problems of the type considered here is difficult at best. Dresher, Karlin, and Shapley [38] suggest a formulation in which a set of nonlinear equations are to be solved, and Dresher and Karlin [54] and Karlin [40] propose a type of fixed-point mapping. Both methods can be exceedingly tedious algebraically even for modest problems, and numerical approximation does not appear to be straightforward.

Any two-person zero-sum static game may be approximated and solved numerically by restraining the players to finite control

sets  $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k\}$  and  $\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_l\}$ , computing the payoff  $b_{ij}$  resulting from the use of  $\underline{u}_i$  by the maximizer and  $\underline{v}_j$  by the minimizer, and then solving the matrix game  $B = \{b_{ij}\}$  for mixtures of the given controls. This brute-force approach tends to obscure any subtleties in the interactions of the players and to be difficult to interpret relative to the given problem. Its sole advantage is its generality.

An alternative solution method, amenable to both numerical approximation and analytic interpretation, may be developed based upon Theorem 4.5. In fact, that theorem implies that we may solve our game problem by solving the following mathematical programming problem:

**Problem:** Find the maximum value of the parameter  $\alpha$   
for which there exists a vector  $\underline{r} \in R$  such that (4.65)  
 $A_\alpha^T \underline{r} \in P_S^*$ , where  $A_\alpha$  is defined by (4.38).

The resulting maximum value of  $\alpha$  is the value  $w$  of the game by Theorem 4.5, the set  $R^0 \subset R$  of all vectors  $\underline{r}^0$  such that  $A_w^T \underline{r}^0 \in P_S^*$  represents the optimal strategies for the maximizer by Theorem 4.3, and separating hyperplanes of  $P_S^*$  and  $S(A, R, w)$  (See Equation 4.36) yield the optimal strategy set  $S^0$  for the minimizer by Theorem 4.4.

For simple problems the constrained maximization problem (4.65) may be solved fairly directly, as is demonstrated in the examples of Chapter 6. For more complicated problems the maximization becomes difficult to visualize geometrically and difficult to manipulate algebraically. Approximation, however, is straight-

forward, for since the sets  $R$  and  $P_S^*$  are convex, they may be approximated by a convex polyhedron and a convex polyhedral cone, respectively, to any desired accuracy; then the constraining sets are polyhedral, and problem (4.65) may be solved as a linear programming problem. This discrete approximation and use of linear programming is the essence of the method which is discussed in some detail in the remainder of this section. One of the examples in Chapter 6 helps illustrate the concepts.

We begin by demonstrating the nature of the linear programming approximation to our problem. Let  $\bar{R}$  be a convex polyhedron and let  $\bar{P}_S^*$  be a convex polyhedral cone. Then the requirement  $\underline{x} \in \bar{R}$  can be expressed by the requirement that  $\underline{x}$  satisfy the linear inequalities.

$$\tilde{r}_i^T \underline{x} \geq 0 \quad i=1, 2, \dots, N_R \quad (4.66)$$

for some finite  $N_R$  and suitable vectors  $\tilde{r}_i$ ; similarly  $\underline{s} \in \bar{P}_S^*$  can be expressed by

$$\tilde{s}_i^T \underline{s} \geq 0 \quad i=1, 2, \dots, N_S \quad (4.67)$$

for a finite integer  $N_S$  and suitable  $\tilde{s}_i$ . Note that we have used our convention  $r_0 = 1$ ,  $\tilde{s}_{i0} = 1$ . Using these representations and the definition of  $A_\alpha$ , we may approximate problem (4.65) by the linear programming problem:

$$\begin{aligned} & \max \alpha \\ & \alpha, \underline{x} \\ & \text{subject to the constraints} \\ & \tilde{r}_i^T \underline{x} \geq 0 \quad i=1, 2, \dots, N_R \end{aligned} \quad (4.68)$$

$$\underline{r}^T A \underline{\tilde{s}}_i - \alpha \geq 0 \quad i=1, 2, \dots, N_S \quad (4.68)$$

This approximation applies to general separable games of the form (4.1), since no special properties of the sets  $R$  and  $P_S^*$  have been utilized.

Creating suitable approximations to  $R$  and to  $P_S^*$  turns out to be straightforward, as each can be handled in either of two ways. By Theorem 4.1,  $R$  is the convex hull of the surface  $C_R$ . If a finite number of points  $\underline{r}_j \in C_R$  are chosen, then  $\bar{R}$  may be formed as the convex hull of those points, and the  $\underline{\tilde{r}}_i$  are then the representations of the hyperplanes defining  $\bar{R}$ . Under these circumstances  $\bar{R} \subset R$ , so that an  $\underline{r}^0$  which is a solution to (4.68) is an admissible moment vector for the maximizer. The value  $\alpha^0$  may, depending upon  $P_S^*$ , tend to underestimate the value  $w$  of the original game.

Forming a convex hull of a given set of points and then finding the defining hyperplanes can be very tedious. If the support hyperplanes of  $R$  are known parametrically, as discussed in Section 4.4, then the  $\underline{\tilde{r}}_i$  for equation (4.68) may be taken as realizations of those hyperplanes for a finite number of parameter choices. By implication  $\bar{R}$  will then be the intersection of the half-spaces defined by those hyperplanes and thus  $R \subset \bar{R}$ . This approximation, while easy to generate, tends to overestimate  $w$ , and it may also produce an optimal strategy vector  $\underline{r}^0 \notin R$ . This latter eventuality requires an additional solution step in order to find  $\underline{r}^* \approx \underline{r}^0$ ,  $\underline{r}^* \in R$ . Note that the vectors  $\underline{\tilde{r}}_i$ ,  $i=1, 2, \dots, N_R$ , represent support hyperplanes to  $\bar{R}$  whether the approximation to  $R$  is internal or external.

This will be useful in establishing optimal c. d. f. 's, as is shown in Section 4. 6.

If the boundary points of  $P_S^*$  are known explicitly, then by forming the convex cone of a finite set of those points and determining the support planes  $\tilde{s}_j$ , we may generate an approximation  $\bar{P}_S^* \subset P_S^*$ . Because of the nature of the interaction of  $\bar{P}_S^*$  and  $R$ ,  $w$  may be underestimated when problem (4. 68) is solved. Also, although the support planes  $\underline{s}$  of  $P_S^*$  belong to  $S$ , the support planes  $\tilde{s}_j$  of  $\bar{P}_S^*$  may not have this property.

An alternative method of creating  $\bar{P}_S^*$  is both simpler and occasionally more useful than the method above. For the purposes of solving the linear programming problem, we are interested only in the support planes to  $\bar{P}_S^*$ . From Theorem 4. 6, the boundary of  $P_S^*$  may be obtained using only the set  $C_S$ . Therefore, we may define a boundary of  $\bar{P}_S^*$  using a finite set of points of  $C_S$ ; i. e., pick  $\tilde{s}_j \in C_S$ ,  $j=1, 2, \dots, N_S$ , for use in (4. 68). This yields  $\bar{P}_S^* \subset P_S^*$  and a possibly overestimated value  $w$ . Since  $\tilde{s}_j \in S$  and  $\tilde{s}_j$  supports  $\bar{P}_S^*$ , if it also supports  $S(A, R, w)$  it will be an approximate optimal strategy for the minimizer.

Because approximations to  $R$  and  $P_S^*$  are reasonably generated and because the game problem may be reduced to a maximization problem, linear programming is clearly a useful tool for approximating the value of a game and the optimum moments for the maximizing player. The strategies for the minimizer, which cannot in general be read off from the solution of (4. 68) because they correspond to separating hyperplanes rather than points, can

be found simply by taking the negative of the original game, so that the minimizer becomes the maximizer. Errors due to approximation can of course be reduced using sophisticated computer programming, fine granularity in the approximations, iterative techniques, and special problem characteristics.

#### 4.6 COMPUTING THE CUMULATIVE DISTRIBUTION FUNCTIONS

The method of dual cones can of course be used to find saddlepoint solutions for given general problems with payoff  $\underline{r}^T A \underline{s}$ , where  $\underline{r}$  and  $\underline{s}$  belong to compact convex sets  $R$  and  $S$ , respectively, but ordinarily such problems are intermediate steps to solving problems with payoff  $J(\underline{u}, \underline{v})$  of the form (4.1), that is, with separable payoff. For these problems it is ultimately desired that optimal cumulative distribution functions (c. d. f. 's)  $F^O(\underline{u})$  and  $G^O(\underline{v})$  be found for the maximizer and minimizer. In this section we consider the problem of finding the function  $F^O(\underline{u})$  corresponding to a given  $\underline{r} \in R$ , with the understanding that the situation for  $G^O(\underline{v})$  and  $\underline{s} \in S$  is analogous.

The fundamental relationship between  $\underline{r}$  and  $F(\underline{u})$  is given by equation (4.7), which in vector form is

$$\underline{r}(F) = \int_U \underline{r}(\underline{u}) dF(\underline{u}) \quad (4.7)$$

where  $\underline{r}(\underline{u})$  results from the defining cost function

$$J(\underline{u}, \underline{v}) = \underline{r}^T(\underline{u}) A \underline{s}(\underline{v}) \quad (4.1)$$

As in Section 4.2, let  $I_{\underline{u}^0}(\underline{u})$  denote the degenerate distribution (4.18) for which the entire probability mass is concentrated at  $\underline{u}^0$ , so that

$$I_{\underline{u}^0}(\underline{u}) = \begin{cases} 1 & \underline{u} \geq \underline{u}^0 \\ 0 & \text{otherwise} \end{cases} \quad (4.69)$$

where the vector inequality denotes element by element inequality.

This distribution has the property, if  $\bar{U}$  is an open set in  $U$ ,

$$\int_{\bar{U}} dI_{\underline{u}^0}(\underline{u}) = \begin{cases} 0 & \underline{u}^0 \notin \bar{U} \subset U \\ 1 & \underline{u}^0 \in \bar{U} \subset U \end{cases} \quad (4.70)$$

Then if  $F(\underline{u})$  is a pure strategy concentrated at  $\underline{u}^0 \in U$ , i. e., if

$F(\underline{u}) = I_{\underline{u}^0}(\underline{u})$ , we have from (4.7) that

$$\underline{r}(F) = \underline{r}(\underline{u}^0) \quad (4.71)$$

Therefore, as can be seen by reviewing the definition (4.13) of the set  $C_R$ , a pure strategy generates a point of  $C_R$ . Furthermore, a point  $\underline{r}^0 \in C_R$  must have at least one inverse point  $\underline{u}^0 \in U$ , implying that there is a  $\underline{u}^0$  such that the pure strategy  $I_{\underline{u}^0}(\underline{u})$  generates  $\underline{r}^0$ .

As stated by Lemma A and used in the  $\underline{u}$  proof of Theorem 4.1, any point  $\underline{r}^0 \in R$  may be written

$$\underline{r}^0 = \sum_{i=1}^{\mu+1} \alpha_i \underline{r}(\underline{u}_i) \quad \alpha_i \geq 0, \sum_{i=1}^{\mu+1} \alpha_i = 1 \quad (4.17)$$

$$\underline{u}_i \in U$$

and this  $\underline{r}^0$  will correspond to a c. d. f.

$$F^0(\underline{u}) = \sum_{i=1}^{\mu+1} \alpha_i I_{\underline{u}_i}(\underline{u}) \quad (4.19)$$

Therefore, any point  $\underline{r}^0 \in R$  may be generated using a c. d. f. which is a finite convex combination of pure strategies. This rather surprising fact is the basis for finding c. d. f. 's, for a general method, given  $\underline{r}^0 \in R$  as a result of the method of dual cones, is to find a convex representation for  $\underline{r}^0$  in terms of points  $\underline{r}_i \in C_R$ ,  $i=1, 2, \dots, k \leq \mu+1$ , and then "invert" the functions  $\underline{r}(\underline{u})$  to find the corresponding pure strategies  $\underline{u}_i$ ,  $i=1, 2, \dots, k$ . The pure strategy set  $\underline{u}_i$ ,  $i=1, 2, \dots, k$  for a c. d. f. is then the spectrum of that c. d. f.

Finding a convex representation of  $\underline{r}^0$  and then inverting the functions  $\underline{r}(\underline{u})$  may be very difficult for some problems, and it is then necessary to try a more direct approach. For example, one might attempt to find the spectrum  $\{\underline{u}_i\}$  and weightings  $\{\alpha_i\}$  as the solution of a programming problem of the type

$$\min \left\| \underline{r}^0 - \sum_{i=1}^{\mu+1} \alpha_i \underline{r}(\underline{u}_i) \right\|^2$$

$\underline{u}_i, \alpha_i \quad i=1, 2, \dots, \mu+1$

subject to constraints (4.72)

$$\sum_{i=1}^{\mu+1} \alpha_i = 1$$

$$\alpha_i \geq 0 \quad i=1, 2, \dots, \mu+1 \quad (\text{Cont'd})$$

$$\begin{aligned} \underline{u}_i \in U \quad i=1, 2, \dots, \mu+1 & \quad (4.72) \\ \text{(i. e., } u_i \in [0, 1], & \\ j=1, 2, \dots, m) & \end{aligned}$$

where the minimum distance is of course zero.

If the functions  $\underline{r}(\underline{u})$  can be inverted, the general approach may be appropriate. The critical part of that approach is to find the spectrum  $\{\underline{u}_i\}$  or the equivalent points  $\underline{r}_i \in C_R$ . The weights  $\alpha_i$  are relatively easy to generate since they appear linearly and must be a solution of

$$\underline{r}^0 = \sum_{i=1}^{\mu+1} \alpha_i \underline{r}_i$$

or

(4.73)

$$\underline{r}^0 = \sum_{i=1}^{\mu+1} \alpha_i \underline{r}(\underline{u}_i)$$

For the special case of scalar controls and polynomial payoffs, Karlin and Shapley [41] show that when

$$r_i(u) = u^i \quad i=1, 2, \dots, \mu$$

and a point  $\underline{r}^0 \in R$  is given, the spectrum of  $\underline{r}^0$  is given by the roots of the polynomial functions generated by determinants of the type (for  $\mu$  even and  $\underline{r}^0$  belonging to the lower surface of  $R$ )

$$\underline{\Delta}_{2m}(t) = \begin{vmatrix} 1 & r_1^0 & r_{m-1}^0 & 1 \\ r_1^0 & r_2^0 & r_m^0 & t \\ r_2^0 & r_3^0 & r_{m+1}^0 & t^2 \\ \vdots & \vdots & \vdots & \vdots \\ r_m^0 & r_{m+1}^0 & r_{2m-1}^0 & t^m \end{vmatrix} \quad (4.74)$$

where  $2m = \mu$ . They also derive other cases. Their results are easily extended to multidimensional uncoupled controls.

Another way to compute the spectrum can be used when the support hyperplanes of  $\hat{R}$  are known. In our discussion we assume that  $\underline{r}^0$  belongs to the boundary of  $R$ , which is for our purposes completely general because the compactness of  $R$  implies that any  $\underline{r} \in R$  may be represented in terms of a convex sum of two boundary points of  $R$ . For any  $\underline{r}^0$  belonging to the boundary of  $\hat{R}$ , we know that there is at least one support hyperplane to  $\hat{R}$  which contains  $\underline{r}^0$ . Let  $\underline{h}^0$  represent such a hyperplane, so that, since  $r_0^0 = 1$  by assumption,

$$\begin{aligned} \underline{h}^0 \text{ }^T \underline{r}^0 &= 0 & \underline{r}^0 &= \begin{bmatrix} 1 \\ \underline{\hat{r}}^0 \end{bmatrix} \\ \underline{h}^0 \text{ }^T \underline{r} &\geq 0 & \text{all } \underline{r} \in R & \end{aligned} \quad (4.75)$$

As already established in Section 4.4 for a less general case, for  $\underline{u}_i$  to belong to the spectrum corresponding to  $\underline{r}^0$ , it is necessary that

$$\underline{h}^{\circ T} \underline{r}(\underline{u}_i) = 0 \quad (4.76)$$

Therefore, we may seek the spectrum among the points  $\underline{r}_j \in C_R$  for which  $\underline{h}^{\circ T} \underline{r}_j = 0$  and find  $\underline{u}_i$  as the inverse of  $\underline{r}_i$ .

An important property of this hyperplane technique is that it is a useful companion to the method of linear programming used to solve the dual cone problem. The solution  $\underline{r}^{\circ}$  of problem (4.68) of necessity lies on at least one face of  $\bar{R}$ , that is, at least one of the inequalities

$$\underline{r}_i^T \underline{r}^{\circ} \geq 0 \quad i=1, 2, \dots, N_R$$

will in fact be an equality for some index  $j$ . But  $\underline{r}_j$  represents a support hyperplane of  $\bar{R}$ . A catalog of the points in  $C_R$  which generate each hyperplane will immediately reveal which such points generate  $\underline{r}_j$  and, by implication, which points belong to a spectrum for  $\underline{r}^{\circ}$ .

#### 4.7 SUMMARY

At this point we take stock of our accomplishments in this chapter. For the static game problem with payoff

$$J(\underline{u}, \underline{v}) = \underline{r}^T(\underline{u}) \underline{A} \underline{s}(\underline{v}) \quad (4.1)$$

where  $\underline{u}$  and  $\underline{v}$  belong to unit hypercubes, we have demonstrated, using the method of dual cones, both a solution technique and an interesting geometrical interpretation of the interactions of the control spaces. Of particular importance are the facts that the game problem was shown to be solvable as a constrained

maximization problem and that approximate numerical solutions are possible using linear programming, for which well-developed computer programs already exist. We also characterized some of the sets involved in special cases and indicated how the optimal c. d. f.'s may be found.

These facts are the foundation for the consideration in Chapter 5 of multistage games.

## CHAPTER 5

### THE SOLUTIONS OF A CLASS OF MULTISTAGE GAMES

In this chapter the problem of finding a saddlepoint for the expected value of the cost function  $J$  of two-person zero-sum  $N$ -stage games of perfect information is discussed. For the games of interest, the cost function has the form

$$J = g_{N+1}(\underline{z}(N+1)) + \sum_{i=1}^N g_i(\underline{z}(i), \underline{u}(i), \underline{v}(i)), \quad (3.3)$$

the dynamics have the form

$$\underline{z}(i+1) = f(\underline{z}(i), \underline{u}(i), \underline{v}(i); i), \quad \underline{z}(1) = \underline{z}, \quad (3.1)$$

and the controls  $\underline{u}(i)$  and  $\underline{v}(i)$  are to be chosen at each stage from  $m$ - and  $n$ -dimensional unit hypercubes  $U$  and  $V$ , respectively. The functions  $g_i$  and  $f$  are polynomials.

Two variations of this dynamic game, that of open loop strategies and that of closed loop strategies, are analyzed using the methods of Chapter 4. This is done by first showing that each of those games can be reduced to certain static games in which the state vector  $\underline{z}$  is a parameter. Then continuity properties of the optimal solutions are established, and finally stronger results for the class of linear-quadratic games are derived. As indicated in Chapter 2, existence of the saddlepoint optimum was established by earlier researchers, who will be cited as appropriate in the next two sections.

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## 5.1 CLOSED-LOOP STRATEGIES AND THE PRINCIPLE OF OPTIMALITY

In Chapter 3 the multistage game with closed loop strategies was defined. The closed loop optimal mixed strategies  $F^0(\underline{u}(i)|\underline{z}(i), i)$  and  $G^0(\underline{v}(i)|\underline{z}(i), i)$  and the value function  $w_i(\underline{z}(i))$  were defined via equation (3.5). By simple substitution in that equation it is clear that the value satisfies the recursive equations

$$\begin{aligned}
 w_{N+1}(\underline{z}) &= g_{N+1}(\underline{z}) \\
 w_i(\underline{z}(i)) &= \int_V \int_U [g_i(\underline{z}(i), \underline{u}, \underline{v}) + w_{i+1}(f(\underline{z}(i), \underline{u}, \underline{v}; i))] \\
 &\quad dF^0(\underline{u}|\underline{z}(i), i) dG^0(\underline{v}|\underline{z}(i), i) \\
 &= \int_{(\underline{u}, \underline{v})}^{V, U} [g_i(\underline{z}(i), \underline{u}, \underline{v}) + w_{i+1}(f(\underline{z}(i), \underline{u}, \underline{v}; i))]
 \end{aligned} \tag{5.1}$$

The fact that such a quantity exists and satisfies (5.1) has been used either explicitly or implicitly by many researchers. Fleming [53] states the necessary facts in a theorem which is directly applicable to the present problem.

Since  $U$  and  $V$  are hypercubes, the problem of solving (5.1) for each  $i$  and for fixed  $\underline{z}(i)$  can be approached by the methods of Chapter 4 provided that the quantity to be optimized is separable in  $\underline{u}$  and  $\underline{v}$ . This is true since by suitable grouping of terms we may write (5.1) as

$$w_i(\underline{z}) = \underset{(\underline{u}, \underline{v})}{\text{val}} \left[ \sum_{k=1}^{\mu} \sum_{j=1}^{\nu} r_k(\underline{u}) a_{kj}(\underline{z}) s_j(\underline{v}) \right] = \underset{(\underline{u}, \underline{v})}{\text{val}} \left[ \underline{r}^T(\underline{u}) A(\underline{z}) \underline{s}(\underline{v}) \right] \quad (5.2)$$

In the special case of polynomials, for example, the functions  $r_k$ ,  $a_{kj}$ ,  $s_j$  have the following forms:

$$r_k(\underline{u}) = u_1^{\xi_{1k}} u_2^{\xi_{2k}} \dots u_m^{\xi_{mk}} \quad k=1, 2, \dots, \mu$$

$$\xi_{ik} = \text{non-negative integer} \quad i=1, \dots, m$$

$$s_j(\underline{v}) = v_1^{\eta_{1j}} v_2^{\eta_{2j}} \dots v_n^{\eta_{nj}} \quad j=1, 2, \dots, \nu \quad (5.3)$$

$$\eta_{kj} = \text{non-negative integer} \quad k=1, \dots, n$$

$$a_{ij}(\underline{z}) = c_{ij} z_1^{\zeta_{1ij}} \dots z_l^{\zeta_{lij}} \quad i=1, 2, \dots, \mu$$

$$j=1, 2, \dots, \nu$$

$$\zeta_{kij} = \text{non-negative integer} \quad k=1, \dots, l$$

This form is analyzed in detail in later sections. Note that it is a parameterized version of the problem of Chapter 4.

The constraint that the right hand side functions in (5.1) be separable is important. The functions  $g_i(\underline{z}, \underline{u}, \underline{v})$  are separable by definition, so it is the term  $w_{i+1}(f(\underline{z}, \underline{u}, \underline{v}; i))$  which is of concern. Unfortunately, as demonstrated in an example in Chapter 6, this term is not always separable. This is not surprising, for even simple optimization problems with parameters often have inflection points which are not of the same form as the given problem. For example, the equation of the maximum in  $t$  of the quadratic function

$$f(z, t) = a_0(z) + a_1(z)t + a_2(z)t^2 \quad a_2(z) < 0$$

is

$$\max_t f(z, t) = a_0(z) - \frac{a_1^2(z)}{4a_2(z)}$$

Although the value function is not always such that  $w_{i+1}(\underline{f}(z, \underline{u}, \underline{v}; i))$  is separable, this term is separable for  $i=N$  and for linear-quadratic problems; the latter fact is proven in Sections 5.4 and 5.5. In addition, it may be separable for other classes of problems and for special regions of problems for which general separability does not hold; this requires further research and detailed analysis of the functions. Finally, for numerical purposes it should be satisfactory to approximate  $w_{i+1}(\underline{f}(z, \underline{u}, \underline{v}; i))$  by a suitable separable function and to apply dynamic programming and the methods of Chapter 4 to the resulting problem.

## 5.2 OPEN-LOOP STRATEGIES AND BATCH PROCESSING SOLUTIONS

In Chapter 3 the polynomial game with open-loop strategies was described. In this section we reduce a stage  $i$  of that game for which  $\underline{z}(i)$  is known to an equivalent single-stage game in which  $\underline{z}(i)$  is a parameter and show that this truncated game may be solved as a batch process. The reduction is essentially algebraic, and the fact that the resulting form is identical to that used in Chapter 4 guarantees a saddlepoint solution.

Without loss of generality, but with a considerable gain in notational convenience, let us consider the problem for  $i=1$ . By

repeatedly substituting (3.1) into (3.3), we may demonstrate explicitly the independent variables in the cost function

(5.4)

$$\begin{aligned}
 J = & g_1(\underline{z}(1), \underline{u}(1), \underline{v}(1)) + g_2(f(\underline{z}(1), \underline{u}(1), \underline{v}(1); 1), \underline{u}(2), \underline{v}(2)) + \dots \\
 & + g_N(f(f(\dots f(\underline{z}(1), \underline{u}(1), \underline{v}(1); 1)\dots), \underline{u}(N-1), \underline{v}(N-1); N-1), \\
 & \quad \underline{u}(N), \underline{v}(N)) \\
 & + g_{N+1}(f(f(\dots f(\underline{z}(1), \underline{u}(1), \underline{v}(1); 1)\dots), \underline{u}(N), \underline{v}(N); N))
 \end{aligned}$$

Because all of the functions  $g_i(\dots)$  and  $f(\dots; i)$  are polynomials in their arguments for all applicable indices  $i$ , this may be rewritten as

$$J = g(\underline{z}(1), \underline{u}(1), \dots, \underline{u}(N), \underline{v}(1), \underline{v}(2), \dots, \underline{v}(N)) \quad (5.5)$$

where  $g$  is a suitable polynomial function of the arguments. We may define an  $mN$ -vector  $\underline{u}$  and an  $nN$ -vector  $\underline{v}$

$$\underline{u} = \begin{bmatrix} \underline{u}(1) \\ \underline{u}(2) \\ \cdot \\ \cdot \\ \cdot \\ \underline{u}(N) \end{bmatrix} \quad \underline{v} = \begin{bmatrix} \underline{v}(1) \\ \underline{v}(2) \\ \cdot \\ \cdot \\ \cdot \\ \underline{v}(N) \end{bmatrix} \quad (5.6)$$

noting that these are elements of an  $mN$ -dimensional unit hypercube  $\underline{U}$  and an  $nN$ -dimensional unit hypercube  $\underline{V}$ , respectively, and rewrite (5.5) as

$$J = g(\underline{z}(1), \underline{u}, \underline{v}) \quad (5.7)$$

where we have simply changed notation and  $g$  is still a polynomial function of the elements of the various vectors. A typical term of (5.7) has the form

$$c \cdot z_1^{\zeta_1} z_2^{\zeta_2} \cdots z_l^{\zeta_l} \cdot u_1^{\xi_1} u_2^{\xi_2} \cdots u_{Nm}^{\xi_{Nm}} \cdot v_1^{\eta_1} v_2^{\eta_2} \cdots v_{Nn}^{\eta_{Nn}}$$

where  $c$  is a constant,  $z_j$  is the  $j^{\text{th}}$  element of  $\underline{z}(1)$ ,  $u_j$  is the  $j^{\text{th}}$  element of  $\underline{u}$ , etc., and all exponents are non-negative finite integers. Define

$$r_0(\underline{u}) \equiv 1$$

$$s_0(\underline{v}) \equiv 1$$

$$r_i(\underline{u}) = u_1^{\xi_1} u_2^{\xi_2} \cdots u_{Nm}^{\xi_{Nm}} \quad (5.8)$$

$$s_j(\underline{v}) = v_1^{\eta_1} v_2^{\eta_2} \cdots v_{Nn}^{\eta_{Nn}}$$

$$a_{ij}(\underline{z}) = c \cdot z_1^{\zeta_1} z_2^{\zeta_2} \cdots z_l^{\zeta_l}$$

where it is implicit that the constant  $c$  and exponents  $\zeta$  depend upon the indices  $i$  and  $j$ , that the exponents  $\xi$  depend upon  $i$ , and that the exponents  $\eta$  depend upon  $j$ . Then we may for suitable finite integers  $\mu$  and  $\nu$  rewrite (5.7) as

$$J = \sum_{j=0}^{\nu} \sum_{i=0}^{\mu} a_{ij}(\underline{z}) r_i(\underline{u}) s_j(\underline{v}) = \underline{r}^T(\underline{u}) A(\underline{z}) \underline{s}(\underline{v}) \quad (5.9)$$

In the vector-matrix representation,  $\underline{r}$  and  $\underline{s}$  are the obvious  $\mu+1$  and  $\nu+1$  dimensional vector functions and  $A$  is a  $(\mu+1) \times (\nu+1)$  matrix function. For a given initial condition  $\underline{z}$ , (5.9) is precisely the

problem which was solved in Chapter 4. It is noteworthy that it is not necessary that the payoff and dynamics functions be polynomials for (5.9) to result from the substitutions of (3.1) into (3.3), although the class of polynomials is perhaps of widest interest to us. Certainly if the functions are separable in  $\underline{z}$ ,  $\underline{u}$ , and  $\underline{v}$  and polynomial in  $\underline{z}$ , the payoff can be written in the separable form (5.9) and solved by the method of dual cones. Many special problems may also have this characteristic.

That (5.9) is equivalent to (3.3) and is solvable by the methods of Chapter 4 is easily shown. The solution of (5.9) is a value  $w$  and a pair of mixed strategies  $F^0(\underline{u}|\underline{z}, 1)$  and  $G^0(\underline{v}|\underline{z}, 1)$ . These are equivalent to the value  $\hat{w}_1(\underline{z})$  and the set of strategies  $\hat{F}_i^0(\underline{u}(i)|\underline{z}, 1; \underline{u}(1), \dots, \underline{u}(i-1))$  and  $\hat{G}_i^0(\underline{v}(i)|\underline{z}, 1; \underline{v}(1), \dots, \underline{v}(i-1))$ , as can be seen by substituting (5.9) into (3.6), changing the order of integration, and grouping terms to get

$$\begin{aligned} \hat{w}_1(\underline{z}) &= \left[ \int_{\underline{U}} \dots \int_{\underline{U}} \underline{r}^T(\underline{u}) d\hat{F}_N^0(\underline{u}(N)|\underline{z}, 1, \underline{u}(1), \dots, \underline{u}(N-1)) \right. \\ &\quad \left. \dots d\hat{F}_1^0(\underline{u}(1)|\underline{z}, 1) \right] \\ &\quad A(\underline{z}) \left[ \int_{\underline{V}} \dots \int_{\underline{V}} \underline{s}(\underline{v}) d\hat{G}_N^0(\underline{v}(N)|\underline{z}, 1; \underline{v}(1), \dots, \underline{v}(N-1)) \right. \\ &\quad \left. \dots d\hat{G}_1^0(\underline{v}(1)|\underline{z}, 1) \right] \\ &= \left[ \int_{\underline{U}} \underline{r}^T(\underline{u}) dF^0(\underline{u}|\underline{z}, 1) \right] A(\underline{z}) \left[ \int_{\underline{V}} \underline{s}(\underline{v}) dG^0(\underline{v}|\underline{z}, 1) \right] \end{aligned}$$

Here  $\underline{U} = U \times U \times \dots \times U$ ,  $\underline{V} = V \times V \times \dots \times V$ . Integrability is no problem since we may restrict the c. d. f. 's used to those with finite spectra if necessary. Hence, solving (5.9) for mixed strategies on  $\underline{u}$  and  $\underline{v}$  is equivalent to solving the open-loop strategy problem, and the former may be done using the method of dual cones.

The mixed strategies  $F^0(\underline{u}|\underline{z}, 1)$  and  $G^0(\underline{v}|\underline{z}, 1)$  have spectra consisting of control histories  $\underline{u}$  and  $\underline{v}$ . If it is necessary to find the optimal mixtures of controls for stage  $i$ , the usual steps of integrating over all admissible controls for the other stages must be performed, a procedure which is reduced to summations because the spectra are finite. During play of a game, when only a realization of the control strategies is needed, this step may be bypassed by choosing a control history  $\underline{u}$  (or  $\underline{v}$ ) in a random manner and then picking out the desired elements  $\underline{u}(i)$  (or  $\underline{v}(i)$ ).

The discussion above applies in a natural manner if the game is assumed to start at stage  $i$  with initial condition  $\underline{z}(1) = \underline{z}$ . Therefore each player will, at any stage for which both obtain new state information  $\underline{z}$ , use the methods of Chapter 4 and the initial condition  $\underline{z}$  to generate his remaining control histories and then select his control for the present stage using a random choice among those histories.

If both players have optimal pure strategies, then the batch processing method may also be used for the game with closed-loop strategies. This fact is discussed in an enlightening manner by Ho [36]. It does not hold when mixed strategies are used, however, as the reader may demonstrate using simple counterexamples.

Example 6.1 is a good one on which to base a counterexample.

### 5.3 CONTINUITY PROPERTIES OF THE SOLUTIONS OF SEPARABLE GAMES

The exact nature of the dependence of the solutions of multistage games on the initial conditions  $\underline{z}$  varies with the structure of the game and must be established on a case by case basis. One particular property, namely continuity, can be shown to hold in fairly general circumstances. We shall discuss continuity conditions for the value function and for the strategies in the present section before moving on to establish sharper results in later portions of this chapter.

Using our previous results and the notation established in Sections 5.1 and 5.2, we know for some polynomial games with closed loop strategies and all with open-loop strategies that the value function  $w(\underline{z})$  satisfies, for given  $\underline{z}$

$$w(\underline{z}) = \min_{\underline{s} \in S} \max_{\underline{r} \in R} \underline{r}^T A(\underline{z}) \underline{s} = \max_{\underline{r} \in R} \min_{\underline{s} \in S} \underline{r}^T A(\underline{z}) \underline{s} \quad (5.10)$$

where  $R$  and  $S$  are convex hulls of continuous mappings of compact sets  $U$  and  $V$  or  $\underline{U}$  and  $\underline{V}$ , respectively. This representation will prove useful in much of the discussion to follow.

The following well-known result is essential to understanding the interactions of the dual cones when the matrix  $A$  is parameterized.

**Theorem 5.1:** If the elements  $a_{ij}(\underline{z})$  of the matrix  $A(\underline{z})$  are continuous in  $\underline{z}$  and if  $R$  and  $S$  are compact, then

$$w(\underline{z}) = \max_{\underline{r} \in R} \min_{\underline{s} \in S} \underline{r}^T A(\underline{z}) \underline{s} \text{ is continuous in } \underline{z}.$$

Proof: Let  $\underline{z}_0$  be an arbitrary element of  $E^k$  and let  $D_\epsilon(\underline{z}_0)$  denote the set such that, for given  $\epsilon > 0$ ,

$$|a_{ij}(\underline{z}) - a_{ij}(\underline{z}_0)| \leq \epsilon, \quad \text{all } i, j$$

for all  $\underline{z} \in D_\epsilon(\underline{z}_0)$ . Such a set exists since the elements of  $A$  are continuous. Then if  $\underline{r}^0(\underline{z})$  and  $\underline{s}^0(\underline{z})$  are optimal moment vectors at  $\underline{z}$ ,

$$\begin{aligned} w(\underline{z}) - w(\underline{z}_0) &= \underline{r}^{0T}(\underline{z}) A(\underline{z}) \underline{s}^0(\underline{z}) - \underline{r}^{0T}(\underline{z}_0) A(\underline{z}_0) \underline{s}^0(\underline{z}_0) \\ &\leq \underline{r}^{0T}(\underline{z}) [A(\underline{z}) - A(\underline{z}_0)] \underline{s}^0(\underline{z}_0) \\ &\leq \epsilon \sum_{i,j} |r_i^0(\underline{z}) s_j^0(\underline{z}_0)| \end{aligned} \quad (5.12)$$

and

$$\begin{aligned} w(\underline{z}) - w(\underline{z}_0) &\geq \underline{r}^{0T}(\underline{z}_0) [A(\underline{z}) - A(\underline{z}_0)] \underline{s}^0(\underline{z}) \\ &\geq -\epsilon \sum_{i,j} |r_i^0(\underline{z}_0) s_j^0(\underline{z})| \end{aligned} \quad (5.13)$$

which, since  $R$  and  $S$  are compact, implies

$$|w(\underline{z}) - w(\underline{z}_0)| \leq k \epsilon \text{ for some finite } k.$$

Another well-known fact is that the limit of the optimal strategies of a sequence of games is an optimal strategy for the limit of the games. This is useful when payoff functions must be approximated, as we shall see in Chapter 6, and for proving results about continuity of optimal strategies. For reference we formalize this fact in the following lemma and indicate a brief proof.

Lemma B: If  $\underline{r}_n, \underline{s}_n$  are optimal strategies for the game  $\underline{r}^T A_{\epsilon_n} \underline{s}$  where  $A_{\epsilon_n}$  is element-by-element within  $\epsilon_n$  of the matrix  $A$

$$|a_{\epsilon_n ij} - a_{ij}| < \epsilon_n$$

and where  $\underline{r}_n$  and  $\underline{s}_n$  must be chosen from compact sets  $R$  and  $S$ , respectively, then there exist limits  $\underline{r}^0$  and  $\underline{s}^0$  of the sequences  $\{\underline{r}_n\}$  and  $\{\underline{s}_n\}$ ,  $\epsilon_n \rightarrow 0$ , which are optimal strategies for the game with matrix  $A$ .

Proof: We indicate the proof for  $\underline{r}^0$ ; analogous results hold for  $\underline{s}^0$ . The existence of limits follows immediately from the fact that  $\{\underline{r}_n\}$  is an infinite sequence in a compact set  $R$  and must therefore have a convergent subsequence with limit point in  $R$ . Call this limit point  $\underline{r}^0$ . Then  $\underline{r}^0$  is an optimal strategy for Player I for the game with matrix  $A$ , for if it were not, then

$$w = \min_{\underline{s} \in S} \max_{\underline{r} \in R} \underline{r}^T A \underline{s} > \min_{\underline{s} \in S} \underline{r}^0{}^T A \underline{s}$$

or

$$|w - \min_{\underline{s} \in S} \underline{r}^0{}^T A \underline{s}| \geq \delta > 0 \quad (5.14)$$

for some  $\delta$ . But if we define

$$w_n = \min_{\underline{s} \in S} \max_{\underline{r} \in R} \underline{r}^T A_{\epsilon_n} \underline{s} = \underline{r}_n^T A_{\epsilon_n} \underline{s}_n \quad (5.15)$$

then (5.14) becomes

$$\begin{aligned} |w - \min_{\underline{s} \in S} \underline{r}^0{}^T A \underline{s}| &\leq |w - w_n| + |w_n - \min_{\underline{s} \in S} \underline{r}_n^T A \underline{s}| \\ &\quad + |\min_{\underline{s} \in S} \underline{r}_n^T A \underline{s} - \min_{\underline{s} \in S} \underline{r}^0{}^T A \underline{s}| \end{aligned} \quad (5.16)$$

The first term on the right can be made less than  $\frac{\epsilon}{3}$  for large enough  $n > N_1$  by the arguments used in Theorem 5.1, which used only the closeness  $\epsilon_n$  of the terms of the matrices  $A(\underline{z})$  and  $A(\underline{z}_n)$ . Similarly the second term can be made less than  $\frac{\epsilon}{3}$  for  $n > N_2$  by arguments using closeness of the matrices and boundedness of  $S$ , and the third term can be made less than  $\frac{\epsilon}{3}$  for  $n > N_3$  using the facts that  $\underline{r}_n \rightarrow \underline{r}^0$  and that  $S$  is compact. Thus

$$|w - \min_{\underline{s} \in S} \underline{r}^0{}^T A \underline{s}| \leq \epsilon \quad (5.17)$$

for arbitrary  $\epsilon > 0$ , contradicting (5.14).

In discussing continuity of moment sets and c. d. f. 's as functions of  $\underline{z}$ , the following version of the definition of upper semicontinuous mappings is useful.

Definition 5-1: A point-to-set mapping  $\psi(\underline{x})$  is called upper semicontinuous at  $\underline{x}_0$  if corresponding to any open set  $\Psi$  containing  $\psi(\underline{x}_0)$  there exists some  $\delta > 0$  such that

$d(\underline{x}, \underline{x}_0) < \delta$  implies  $\psi(\underline{x}) \in \Psi$  where  $d(\cdot, \cdot)$  is a metric defined on the domain of  $\psi$ .

Using this definition, we adapt a theorem of Karlin [40] to our interests.

**Theorem 5.2:** The set  $R^0(\underline{z})$  of optimal strategies for the game defined by  $\underline{r}^T A(\underline{z}) \underline{s}$ ,  $\underline{r} \in R$ ,  $\underline{s} \in S$ , is an upper-semicontinuous function of the parameter  $\underline{z}$ .

**Proof:** Let  $\underline{z}_0$  be an arbitrary point in  $E^k$  and let  $R^0(\underline{z}_0)$  be the optimal moments for the game with initial condition  $\underline{z}_0$ . Suppose  $H$  is an arbitrary open set such that  $R^0(\underline{z}_0) \subset H$ . Let  $D_\epsilon(\underline{z}_0)$  be as in the proof of Theorem 5.1 and let  $R_\epsilon$  be the set of all moments  $\underline{r} \in R$  which are optimal for at least one  $\underline{z} \in D_\epsilon(\underline{z}_0)$ . We must show that for  $\epsilon \rightarrow 0$  sufficiently small,  $R_\epsilon \subset H$ .

Suppose the contrary. Then there exists a sequence  $\{\epsilon_n\}$ ,  $\epsilon_n \rightarrow 0$ , such that  $R_{\epsilon_n} \not\subset H$  for all  $n$ . For each  $n$ , choose  $\underline{z}_n \in D_{\epsilon_n}(\underline{z}_0)$  with corresponding optimal strategy  $\underline{r}_n$  such that  $\underline{r}_n \notin H$ . Then we have a sequence  $\{\underline{r}_n\}$  in a compact set  $R$  such that  $\underline{r}_n \notin H$ . Thus the sequence will have a convergent subsequence with some limit point  $\underline{r}^0 \in R$ ,  $\underline{r}^0 \notin H$ . But by Lemma B,  $\underline{r}^0$  is an optimal moment vector for the game  $\underline{r}^T A(\underline{z}_0) \underline{s}$ , and therefore  $\underline{r}^0 \in R^0(\underline{z}_0) \subset H$ , a contradiction which completes the proof.

At this point we go beyond previous work to establish a form of continuity for the optimal cumulative distribution functions  $F^0(\underline{u}|\underline{z})$  and  $G^0(\underline{v}|\underline{z})$ . Some of the pitfalls are recognizable in advance and must be carefully circumvented. In particular, we must remember that (1) the optimal c. d. f. 's are not necessarily unique, and (2) the c. d. f. 's are discrete over the sets  $\underline{U}$  and  $\underline{V}$ , and hence continuity in  $\underline{z}$  is much like the continuity of the zeros of a polynomial as functions of the coefficients.

We shall develop our theory using the support hyperplanes to  $R$  at its boundary points. We remember that by assumption  $r_0 = 1$  for  $\underline{r} \in R$ , and without loss of generality we assume that bounded normals of hyperplanes have length less than or equal to unity.

**Theorem 5.3:** The set  $H(\underline{r})$  of the bounded representations (i. e., normals) of the support hyperplanes to  $\hat{R}$  at  $\hat{\underline{r}}$  is an upper semicontinuous function of the boundary points of  $\hat{R}$ .

**Proof:** Let  $\hat{\underline{r}}^0$  belong to the boundary of  $\hat{R}$ , let  $H(\underline{r}^0)$  be the set of all  $\underline{h}$  such that  $\underline{h}^T \underline{r}^0 = 0$ ,  $\underline{h}^T \underline{r} \geq 0$  for all  $\underline{r} \in R$ , and  $\|\underline{h}\| \leq 1$  where  $\underline{r}^0 = \begin{bmatrix} 1 \\ \hat{\underline{r}}^0 \end{bmatrix}$ , and let  $\tilde{H}$  be an open set containing  $H(\underline{r}^0)$ . We assume that the contrary of the theorem holds and that  $D_\epsilon$  is the open set of all  $\underline{r}$  in the boundary of  $R$  such that  $\|\underline{r} - \underline{r}^0\| < \epsilon$ . Then if  $\{\epsilon_n\}$  is a real sequence,  $\epsilon_n > 0$ ,  $\epsilon_n \rightarrow 0$ , we have that  $\underline{r}_n \in D_{\epsilon_n}$  has limit point  $\underline{r}^0$ . Furthermore, if  $H_n$  is the set of all  $\underline{h}$  which

support  $\hat{R}$  at at least one point of  $D_{\epsilon_n}$ , we have  $H_n \notin \tilde{H}$  as our contrary assumption. The set of all hyperplanes with normals of unity or less is necessarily compact for the compact convex set  $\hat{R}$  and in fact is a portion of the dual cone  $P_R^*$ . Choose from each  $H_n$  a vector  $\underline{h}_n \in \tilde{H}$ . Then the sequence  $\{\underline{h}_n\}$  has a limit point, call it  $\underline{h}^0$ , such that  $\underline{h}^0 \in \tilde{H}$ . But  $\underline{h}^0$  supports  $\hat{R}$ , and thus  $R$ . Thus we must have

$$\underline{h}^0 \text{ }^T \underline{r}^0 \geq \delta > 0$$

Since  $\underline{h}_n \text{ }^T \underline{r}_n = 0$  for some  $\underline{r}_n \in D_{\epsilon_n}$  for each  $n$ , we have

$$\underline{h}^0 \text{ }^T \underline{r}^0 - \underline{h}_n \text{ }^T \underline{r}_n \geq \delta > 0$$

But

$$\begin{aligned} |\underline{h}^0 \text{ }^T \underline{r}^0 - \underline{h}_n \text{ }^T \underline{r}_n| &= |\underline{h}^0 \text{ }^T (\underline{r}^0 - \underline{r}_n) - (\underline{h}_n - \underline{h}^0) \text{ }^T \underline{r}_n| \\ &\leq \|\underline{h}^0\| \cdot \|\underline{r}^0 - \underline{r}_n\| + \|\underline{h}_n - \underline{h}^0\| \cdot \|\underline{r}_n\| \end{aligned} \tag{5.18}$$

which can be made arbitrarily small because  $\underline{r}_n \rightarrow \underline{r}^0$  and  $\underline{h}_n \rightarrow \underline{h}^0$ , a contradiction which completes our proof.

Corollary  
5.3-1:

The set  $H'(\underline{z})$  of the bounded representations of the support hyperplanes to  $\hat{R}$  at the optimal strategies  $R^0(\underline{z})$  of the game with initial condition  $\underline{z}$  is an upper

semicontinuous function of  $\underline{z}$  provided  $R^{\circ}(\underline{z})$  consists of boundary points of  $R$ .

**Proof:** This follows immediately from Theorem 5.3 by using Theorem 5.2 and the definition of  $H^{\circ}(\underline{z})$ .

Our next theorem leads to a characterization of the continuity of the spectrum of  $F^{\circ}(\underline{u}|\underline{z})$ .

**Theorem 5.4:** The set  $\varphi(\underline{h})$  of all contact points of the support hyperplane to  $\hat{R}$  represented by  $\underline{h}$  with the set  $C_R$  is an upper semicontinuous function of  $\underline{h}$ .

**Proof:** We remember that  $R$  is the convex hull of  $C_R$ . Also we remark that  $\varphi$  may or may not be connected. We proceed much as in the proofs above, taking a sequence  $\underline{h}_n$  of normals to support hyperplanes to  $\hat{R}$  and looking at their sets  $\varphi_n$  of contact points with  $C_R$ . If  $\underline{h}^{\circ}$  is the limit of  $\underline{h}_n$  but no  $\varphi_n$  is contained in the open set  $\tilde{\varphi}$  which contains  $\varphi(\underline{h}^{\circ})$ , then there must be a sequence of points  $\underline{r}_n \in C_R$ ,  $\underline{r}_n \notin \varphi(\underline{h}^{\circ})$ , such that  $\underline{r}_n \rightarrow \underline{r}^{\circ} \notin \varphi(\underline{h}^{\circ})$ . But  $\varphi(\underline{h}^{\circ})$  is the set

$$\varphi(\underline{h}^{\circ}) = \{\underline{r} | \underline{r} \in C_R, \underline{h}^{\circ T} \underline{r} = 0\}$$

and thus, since  $\underline{h}^{\circ}$  supports  $\hat{R}$ , we must have

$$\underline{h}^{\circ T} \underline{r}^{\circ} \geq \delta > 0$$

for some  $\delta$ . This situation is similar to that of

semicontinuous function of  $\underline{z}$  provided  $R^\circ(\underline{z})$  consists of boundary points of  $R$ .

**Proof:** This follows immediately from Theorem 5.3 by using Theorem 5.2 and the definition of  $H'(\underline{z})$ .

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**Proof:** We remember that  $R$  is the convex hull of  $C_R$ . Also we remark that  $\varphi$  may or may not be connected. We proceed much as in the proofs above, taking a sequence  $\underline{h}_n$  of normals to support hyperplanes to  $\hat{R}$  and looking at their sets  $\varphi_n$  of contact points with  $C_R$ . If  $\underline{h}^\circ$  is the limit of  $\underline{h}_n$  but no  $\varphi_n$  is contained in the open set  $\tilde{\varphi}$  which contains  $\varphi(\underline{h}^\circ)$ , then there must be a sequence of points  $\underline{r}_n \in C_R$ ,  $\underline{r}_n \notin \varphi(\underline{h}^\circ)$ , such that  $\underline{r}_n \rightarrow \underline{r}^\circ \notin \varphi(\underline{h}^\circ)$ . But  $\varphi(\underline{h}^\circ)$  is the set

$$\varphi(\underline{h}^\circ) = \{\underline{r} | \underline{r} \in C_R, \underline{h}^{\circ T} \underline{r} = 0\}$$

and thus, since  $\underline{h}^\circ$  supports  $\hat{R}$ , we must have

$$\underline{h}^{\circ T} \underline{r}^\circ \geq \delta > 0$$

for some  $\delta$ . This situation is similar to that of

Theorem 5.3 and in particular to equation (5.18), and a similar contradiction arises, completing the proof.

Corollary  
5.4-1:

The set  $\varphi'(\underline{x})$  of all contact points of all support hyperplanes to  $\hat{R}$  at  $\underline{x}$  with the set  $C_R$  is an upper semicontinuous function of  $\underline{x}$ .

Corollary  
5.4-2:

The set  $\varphi''(\underline{z})$  of all contact points of all support hyperplanes to  $\hat{R}$  at points  $\underline{x} \in R^0(\underline{z})$  with the set  $C_R$  is an upper semicontinuous function of  $\underline{z}$ , provided that  $R^0(\underline{z})$  consists only of boundary points of  $R$ .

We remark that Hurwitz's theorem gives a version of these results for the special case of zeros of polynomials as functions of their coefficients. For the game problem this is similar to a case with polynomial functions and scalar controls. Note that the corollaries to Theorem 5.4 require that all support hyperplanes of the given class be considered.

There is one more step before establishing our final result. We remember that Lemma A implies that for  $\underline{x} \in R$  it is possible to form a finite convex representation for  $\underline{x}$  in terms of elements  $\underline{x}_i \in C_R$ , where  $R$  is the convex hull of  $C_R$ . We may write such a representation as

$$\underline{r} = \sum_{i=1}^{\mu+1} \alpha_i \underline{r}_i \quad \alpha_i \geq 0, \underline{r}_i \in C_R \quad i=1, 2, \dots, \mu+1$$

$$\sum_{i=1}^{\mu+1} \alpha_i = 1$$

We are interested in establishing continuity for the convex coefficients  $\alpha_i$ . Each coefficient is a function of the vector  $\underline{r}$  being represented, of the spectrum  $\underline{r}_1, \underline{r}_2, \dots, \underline{r}_{\mu+1}$  used, and of the index  $i$ . Thus when the representation of  $\underline{r}$  is not unique or when a set of vectors  $\underline{r}$  is to be represented, one becomes concerned with an infinite set of such functions  $\alpha_i$ . Fortunately, our purposes are served by a more modest theorem than one concerning continuity of this set.

**Theorem 5.5:** If a sequence  $\{\underline{r}(n)\}$  has limit  $\underline{r}^0$ , if each  $\underline{r}(n)$  has convex representation  $\sum_{i=1}^{\mu+1} \alpha_i(n) \underline{r}_i(n)$ , then  $\underline{r}^0$  has representation  $\sum_{i=1}^{\mu+1} \alpha_i^0 \underline{r}_i^0$  where  $\underline{r}_i(n) \rightarrow \underline{r}_i^0$  and  $\alpha_i(n) \rightarrow \alpha_i^0$  for each  $i$ .

**Proof:** Since each  $\alpha_i(n) \in [0, 1]$  and each  $\underline{r}_i(n) \in C_R$ , both of which are compact sets, each sequence has a convergent subsequence. (We assume implicitly that the elements are kept ordered, so that the limits will be independent.) Denote the limits by  $\alpha_i^0$  and  $\underline{r}_i^0$ . We are to show that  $\underline{r}^0 = \sum_{i=1}^{\mu+1} \alpha_i^0 \underline{r}_i^0$ . Suppose the contrary. Then

$$\|\underline{r}^0 - \sum_{i=1}^{\mu+1} \alpha_i^0 \underline{r}_i^0\| \geq \delta > 0 \quad (5.19)$$

But

$$\begin{aligned} \|\underline{r}^0 - \sum_{i=1}^{\mu+1} \alpha_i^0 \underline{r}_i^0\| &= \|\underline{r}^0 - \underline{r}(n) + \sum_{i=1}^{\mu+1} (\alpha_i(n) \underline{r}_i(n) - \alpha_i^0 \underline{r}_i^0)\| \\ &\leq \|\underline{r}^0 - \underline{r}(n)\| + \sum_{i=1}^{\mu+1} (\alpha_i(n) \|\underline{r}_i(n) - \underline{r}_i^0\| + \\ &\quad + \|\underline{r}_i^0\| |\alpha_i(n) - \alpha_i^0|) \\ &\leq \epsilon_1 + \sum_{i=1}^{\mu+1} (\alpha_i(n) \epsilon_2 + \|\underline{r}_i^0\| \epsilon_3) \end{aligned}$$

for sufficiently large  $n$  and arbitrary positive  $\epsilon_1$ ,  $\epsilon_2$ ,  $\epsilon_3$ . Since  $\alpha_i(n)$  and  $\|\underline{r}_i^0\|$  are bounded, this contradicts (5.19) and completes our proof.

Using this theorem, we are able to develop a statement of a form of continuity for the c. d. f. 's in terms of the initial condition  $\underline{z}$  of the state vector. To do this, we extend the concept of spectrum of a c. d. f. slightly by defining generalized spectrum sets.

Let  $R^0(\underline{z})$  be the set of optimal moments for the maximizer for the game starting at  $\underline{z}$ . Then an element  $\underline{u}$  of  $U$  is said to belong to the generalized spectrum at  $\underline{z}$  if there exists a convex representation of some  $\underline{r}^0 \in R^0(\underline{z})$  in terms of boundary points of  $R$  such that at least one support hyperplane to  $\hat{R}$  at one of these boundary

points contains a point  $\underline{f} \in C_R$  which is the image of  $\underline{u}$  under the transformations  $\underline{r}(\underline{u})$ . From the discussion of Section 4.6 relating c. d. f. 's to moment vectors, it follows that the spectrum of any optimal c. d. f. for the maximizer at  $\underline{z}$  is contained in the generalized spectrum. The generalized spectrum thus contains all vectors  $\underline{u}$  which might belong to a spectrum of an optimal c. d. f. at  $\underline{z}$ . A generalized spectrum for the minimizer may be defined analogously.

Using the definition of generalized spectrum and the results of Corollary 5.4-2 and Theorem 5.5, it is little more than a restatement of those results to obtain the following important theorem.

**Theorem 5.6:** The generalized spectrum for each player is an upper semicontinuous function of  $\underline{z}$ . For given spectrum elements in this set, the corresponding weightings are also upper semicontinuous in  $\underline{z}$ .

Loosely put, the implications of Theorem 5.6 are that if  $\underline{z} \rightarrow \underline{z}_0$ , then in an upper semicontinuous sense

$$\begin{aligned}
 F^0(\underline{u} | \underline{z}) &= \sum_{i=1}^{\mu+1} \alpha_i(\underline{z}) I_{\underline{u}_i(\underline{z})}(\underline{u}) \rightarrow F^0(\underline{u} | \underline{z}_0) = \\
 &= \sum_{i=1}^{\mu+1} \alpha_i(\underline{z}_0) I_{\underline{u}_i(\underline{z}_0)}(\underline{u})
 \end{aligned}
 \tag{5.20}$$

The upper semicontinuity is required primarily because of lack of uniqueness of solutions. The use of generalized spectra allows for the case in which  $\alpha_i(\underline{z}) \rightarrow 0$  as  $\underline{z} \rightarrow \underline{z}_0$ , since our definition of

spectrum would not then consider  $\underline{u}_i(\underline{z}_0)$  as a spectrum point of  $F^0(\underline{u}|\underline{z}_0)$ .

These concepts of continuity are important in understanding the effects of parameterization of the solutions introduced by considering dynamic games. The continuity of the value function and upper semicontinuity of the optimal moment sets are particularly useful in visualizing those effects and in problem solving.

#### 5.4 A LINEAR QUADRATIC GAME WITH SCALAR CONTROLS

In this section it is demonstrated that the value function for one special class of games is piecewise polynomial and therefore that the principle of optimality may be applied along with the parameterized method of dual cones in order to find a solution. In the course of the demonstration, the use of Theorem 4.5 is shown, the solution to the problem is generated, and the ideas to be used in the more general problem of the next section are illustrated.

Let  $\underline{z}(i)$  be an  $l$ -vector with stage index  $i$  which satisfies

$$\underline{z}(i+1) = T_i \underline{z}(i) + \underline{\alpha}_i u(i) + \underline{\beta}_i v(i) + \underline{\gamma}_i \quad (5.21)$$

where  $T_i$  is an  $l \times l$  matrix,  $\underline{\alpha}_i$  and  $\underline{\beta}_i$  are  $l$ -vectors,  $u(i)$  and  $v(i)$  are scalars to be chosen from the unit interval  $[0, 1]$ , and  $\underline{\gamma}_i$  is an  $l$ -vector. Player  $J$  is to choose mixed strategies  $F_i^0(u|\underline{z})$  to maximize the minimum expected value of the payoff function

$$J = \underline{z}^T(N+1) \underline{c}_{N+1}^* \underline{z}(N+1) + \underline{e}_{N+1}^T \underline{z}(N+1) + \epsilon_{N+1} \quad (5.22)$$

(Cont'd)

$$\begin{aligned}
& + \sum_{i=1}^N (\underline{z}^T(i) \underline{C}_i \underline{z}(i) + \underline{e}_i^T \underline{z}(i) + \underline{\Delta}_i^T \underline{z}(i) u(i) + \underline{\xi}_i^T \underline{z}(i) v(i)) \\
& + P_i u^2(i) + p_i u(i) + \rho_i u(i) v(i) + Q_i v^2(i) + q_i v(i)
\end{aligned} \tag{5.22}$$

where the  $l \times l$  matrices  $\underline{C}_i$ , the  $l$ -vectors  $\underline{e}_i$ ,  $\underline{\Delta}_i$ ,  $\underline{\xi}_i$ , and the scalars  $P_i$ ,  $p_i$ ,  $Q_i$ ,  $q_i$ ,  $\rho_i$ ,  $\epsilon_{N+1}$  are known to both players.

Player II will choose mixed strategies  $G_i^0(v|\underline{z})$  to minimize the maximum expected value. A special case of this problem is given in great detail as the first example of Chapter 6, so the argument below is somewhat abbreviated.

We proceed by induction on the indices  $i$ , taken in reverse order. Define  $w_i(\underline{z})$ ,

$$\begin{aligned}
w_i(\underline{z}) = \text{val}_{(u(i), v(i))} & \left[ \underline{z}^T \underline{C}_i \underline{z} + \underline{e}_i^T \underline{z} + P_i u^2(i) + p_i u(i) + \rho_i u(i) v(i) \right. \\
& + \underline{\Delta}_i^T \underline{z} u(i) + \underline{\xi}_i^T \underline{z} v(i) + Q_i v^2(i) + q_i v(i) \\
& \left. + w_{i+1}(\underline{T}_i \underline{z} + \underline{\alpha}_i u(i) + \underline{\beta}_i v(i) + \underline{\gamma}_i) \right] \\
& i=1, 2, \dots, N
\end{aligned} \tag{5.23}$$

in the usual manner, and note that

$$w_{N+1}(\underline{z}) = \underline{z}^T \underline{C}_{N+1} \underline{z} + \underline{e}_{N+1}^T \underline{z} + \epsilon_{N+1} \tag{5.24}$$

We assume that  $w_{i+1}(\underline{z})$  is quadratic in  $\underline{z}$  in some region of interest, i. e.,

$$w_{i+1}(\underline{z}) = \underline{z}^T D_{i+1} \underline{z} + \underline{d}_{i+1}^T \underline{z} + \delta_{i+1} \quad (5.25)$$

and attempt to prove that  $w_i(\underline{z})$  is piecewise quadratic, that is, that  $w_i(\underline{z})$  is given by some quadratic form in  $\underline{z}$  for any region of  $E^k$ .

Let us make the following definitions

$$\begin{aligned} D &= T_i^T D_{i+1} T_i + \underline{C}_i \\ \underline{d} &= T_i^T (\underline{d}_{i+1} + 2D_{i+1} \underline{\gamma}_i^T) + \underline{e}_i \\ P &= \underline{\alpha}_i^T D_{i+1} \underline{\alpha}_i + P_i \\ p &= 2\underline{\alpha}_i^T D_{i+1} \underline{\gamma}_i + \underline{d}_{i+1}^T \underline{\alpha}_i + P_i \\ Q &= \underline{\beta}_i^T D_{i+1} \underline{\beta}_i + Q_i \\ q &= 2\underline{\beta}_i^T D_{i+1} \underline{\gamma}_i + \underline{d}_{i+1}^T \underline{\beta}_i + q_i \\ \underline{\Delta} &= \underline{\Delta}_i + 2T_i^T D_{i+1} \underline{\alpha}_i \\ \underline{\xi} &= \underline{\xi}_i + 2T_i^T D_{i+1} \underline{\beta}_i \\ \rho &= \rho_i + 2\underline{\alpha}_i^T D_{i+1} \underline{\beta}_i \\ \delta &= \delta_{i+1} + \underline{\gamma}_i^T D_{i+1} \underline{\gamma}_i + \underline{d}_{i+1}^T \underline{\gamma}_i \end{aligned} \quad (5.26)$$

Using these definitions, along with the assumption that  $E_{i+1}$  is symmetric, (5.23) becomes, under assumption (5.25)

$$\begin{aligned} w_i(\underline{z}) = \text{val}_{(u, v)} [ &\underline{z}^T D \underline{z} + \underline{d}^T \underline{z} + Pu^2 + Qv^2 + pu + qv + \rho uv \\ &+ \delta + \underline{\Delta}^T \underline{z}u + \underline{\xi}^T \underline{z}v] \end{aligned} \quad (5.27)$$

We may write this in vector matrix form.

$$w_1(\underline{z}) = \text{val}_{(u, v)} \left\{ [1 \ u \ u^2] \begin{bmatrix} \underline{z}^T \underline{D} \underline{z} + \underline{d}^T \underline{z} + \delta & \underline{\xi}^T \underline{z} + q & Q \\ \underline{\Delta}^T \underline{z} + P & \rho & 0 \\ P & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} \right\} \quad (5.28)$$

Defining

$$A(\underline{z}) = \begin{bmatrix} \underline{z}^T \underline{D} \underline{z} + \underline{d}^T \underline{z} + \delta & \underline{\xi}^T \underline{z} + q & Q \\ \underline{\Delta}^T \underline{z} + P & \rho & 0 \\ P & 0 & 0 \end{bmatrix}$$

$$\underline{r} = \int_0^1 \begin{bmatrix} 1 \\ u \\ u^2 \end{bmatrix} dF(u) \quad (5.29)$$

$$\underline{s} = \int_0^1 \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} dG(v)$$

let us write (5.28) as

$$w_1(\underline{z}) = \max_{\underline{r} \in R} \min_{\underline{s} \in S} \{ \underline{r}^T A(\underline{z}) \underline{s} \}$$

As developed in Section 5.1, for this problem  $C_R$  is the set  $\{ \underline{r} | r_0 = 1, r_1 = t, r_2 = t^2, t \in [0, 1] \}$ ,  $R$  is the convex hull of  $C_R$  and is the region enclosed by

$$r_0 = 1; r_2 \leq r_1 \text{ and } r_2 \geq r_1^2 \quad (5.30)$$

or parametrically by the curve  $C_R$  and by  $C' = \{\underline{r} | r_0 = 1, r_1 = r_2 = t, t \in [0, 1]\}$ .  $S$  is analogous to  $R$ . The dual cone  $P_S^*$  is defined by the boundary curves

$$\begin{aligned} \text{a. } & s_0 = 0 \text{ for } s_1 \geq 0 \text{ and } s_1 > -s_2 \\ \text{b. } & s_0 + s_1 + s_2 = 0 \text{ for } s_0 \geq 0, s_1 \leq -2s_2 \\ \text{c. } & 4s_0s_2 - s_1^2 = 0 \text{ for } s_2 \geq 0, 0 \geq s_1 \geq -2s_2 \end{aligned} \quad (5.31)$$

The sets  $\hat{R}$ , which is the projection of  $R$  on the  $(r_1, r_2)$  plane, and  $P_S^*$  are sketched in Figures 5-1 and 5-2. The set  $P_S^*$  is the space in the positive  $s_0$ -direction with boundary given by (5.31).

At this point we introduce a parameter  $\alpha$ , to be maximized according to the dictates of Theorem 4.5. The elements of the set  $S(A, R, \alpha)$  defined in equation (4.36) are then given by vectors  $\underline{s} \in E^3$  such that

$$\begin{aligned} s_0 &= \underline{z}^T D \underline{z} + \underline{d}^T \underline{z} + \delta - \alpha + (\underline{\Delta}^T \underline{z} + p)r_1 + P r_2 \\ s_1 &= \underline{\xi}^T \underline{z} + q + \rho r_1 \\ s_2 &= \Omega \end{aligned} \quad (5.32)$$

Since  $A(\underline{z})$  is linear for any  $\underline{z}$ , its boundaries correspond to those of  $R$ . Trivial or special cases such as  $P = 0$  will not overly concern us, since the methods below still apply.

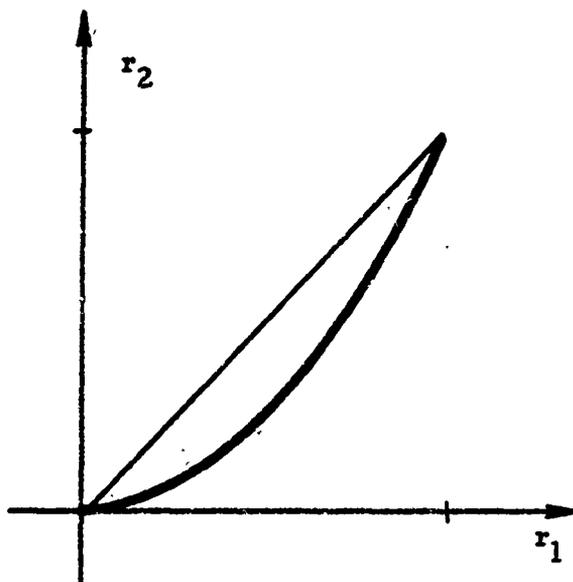


Figure 5-1. The set  $\hat{R}$ .

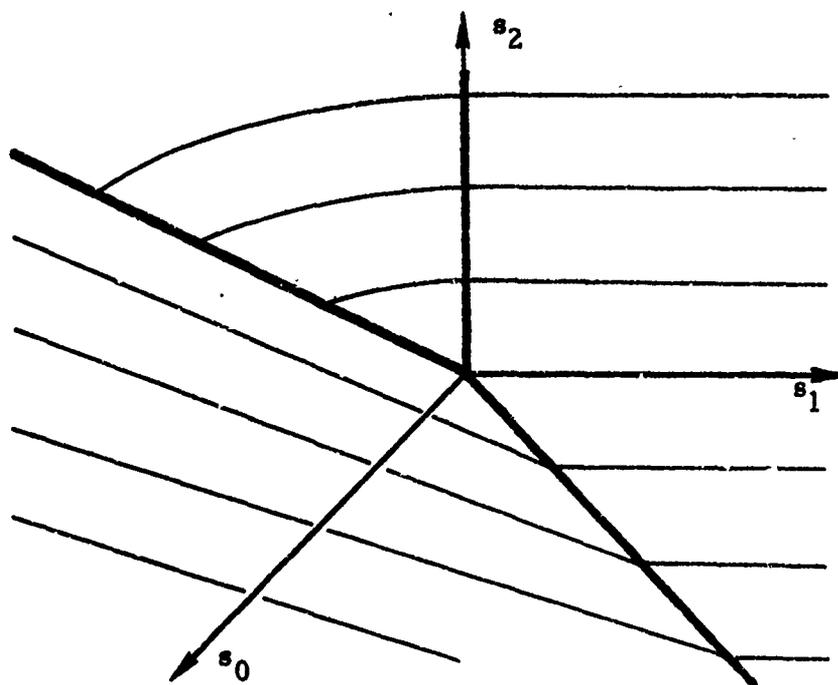


Figure 5-2. The set  $P_S^*$ .

The implications of the continuity proofs of Section 5.3 are that  $S(A(\underline{z}), R, w_i(\underline{z}))$  moves smoothly over  $P_S^*$  as  $\underline{z}$  is varied. This essential fact may be verified here by substitution of numerical values, and is used in the discussion below. Basically, if  $P_S^*$  and  $S(A(\underline{z}), R, w_i(\underline{z}))$  are in contact at a point which is internal to one of the identifiable boundary regions of each, then as  $\underline{z}$  varies slightly, these surfaces will remain in contact although the exact contact point may move. Therefore, we may restrict our attention to pairs of surfaces, one each from  $S(A(\underline{z}), R, w_i(\underline{z}))$  and  $P_S^*$ , in generating  $w_i(\underline{z})$ . We will simply examine the possibilities exhaustively, using the curves

$$\begin{aligned}
 \text{a. } r_2 &= r_1^2 \\
 &\text{and} \\
 \text{b. } r_2 &= r_1
 \end{aligned}
 \tag{5.33}$$

of  $\hat{R}$  and the surfaces and edge

$$\begin{aligned}
 \text{a. } s_0 &= 0 \\
 \text{b. } s_0 + s_1 + s_2 &= 0 \\
 \text{c. } s_0 = 0, s_1 + s_2 &= 0 \\
 \text{d. } 4s_0s_2 - s_1^2 &= 0
 \end{aligned}
 \tag{5.34}$$

of  $P_S^*$ . We shall find the value  $w_i(\underline{z})$  and the optimal mixed strategies  $F^0(u|\underline{z})$  and  $G^0(v|\underline{z})$  for each possibility. Where strategies are not unique, we shall simply demonstrate at least one optimal strategy.

Case 1. The plane  $s_0 = 0$  of  $P_S^*$ . All support hyperplanes to this surface except at the line  $s_1 + s_2 = 0$  (Case 3) have representations  $(\lambda, 0, 0)^T$ ,  $\lambda > 0$ . This implies the moments  $(1 \ 0 \ 0)^T$  for the minimizer with corresponding pure strategy

$$G^0(v|\underline{z}) = I_0(v) \quad (5.35)$$

From (5.32) we immediately have, using (5.33)

$$a. \quad \alpha = \underline{z}^T D \underline{z} + \underline{d}^T \underline{z} + \delta + (\underline{\Delta}^T \underline{z} + p)r_1 + Pr_1^2 \quad (5.36)$$

$$b. \quad \alpha = \underline{z}^T D \underline{z} + \underline{d}^T \underline{z} + \delta + (\underline{\Delta}^T \underline{z} + p + P)r_1$$

where  $r_1 \in [0, 1]$ . Examination of coefficients and maximization of  $\alpha$  over  $r_1$  leads to the following results. (5.37)

i.  $P \geq 0$ ,  $\underline{\Delta}^T \underline{z} + p + P \leq 0$ . Then  $r_1^0 = 0$ ,  $F^0(u|\underline{z}) = I_0(u)$ ,  
and  $w_1(\underline{z}) = \alpha_{\max} = \underline{z}^T D \underline{z} + \underline{d}^T \underline{z} + \delta$

ii.  $P \geq 0$ ,  $\underline{\Delta}^T \underline{z} + p + P \geq 0$ . Then  $r_1^0 = 1$ ,  $F^0(u|\underline{z}) = I_1(u)$ ,  
and  $w_1(\underline{z}) = \underline{z}^T D \underline{z} + (\underline{d} + \underline{\Delta})^T \underline{z} + \delta + p + P$

iii.  $P < 0$ ,  $0 \leq -(\underline{\Delta}^T \underline{z} + p)/2P \leq 1$ . Then  $r_1^0 = -(\underline{\Delta}^T \underline{z} + p)/2P$   
and  $F^0(u|\underline{z}) = I_{r_1^0}(u)$ . Also, the value is  
$$w_1(\underline{z}) = \underline{z}^T \left( D - \frac{\underline{\Delta} \underline{\Delta}^T}{4P} \right) \underline{z} + \left( \underline{d} - \frac{p \underline{\Delta}}{2P} \right)^T \underline{z} + \delta - \frac{p^2}{4P}$$

(Cont'd)

iv.  $P < 0, -(\underline{\Delta}^T \underline{z} + p)/2P \leq 0$ . Same result as i. (5.37)

v.  $P < 0, -(\underline{\Delta}^T \underline{z} + p)/2P \geq 1$ . Same result as ii.

Case 2. The plane  $s_0 + s_1 + s_2 = 0$  of  $P_S^*$ . In this case we find find that  $\underline{s}^0 = (1 \ 1 \ 1)^T$  so that  $G^0(v|\underline{z}) = I_1(v)$ . Substituting (5.32) in (5.34b) and using (5.33) gives (5.38)

a.  $\alpha = \underline{z}^T D \underline{z} + (\underline{d} + \underline{\xi})^T \underline{z} + \delta + q + Q + (\underline{\Delta}^T \underline{z} + p + \rho)r_1 + Pr_1^2$

b.  $\alpha = \underline{z}^T D \underline{z} + (\underline{d} + \underline{\xi})^T \underline{z} + \delta + q + Q + (\underline{\Delta}^T \underline{z} + p + \rho + P)r_1$

Once again we maximize  $\alpha$  over  $r_1 \in [0, 1]$  to get the following results. (5.39)

i.  $P \geq 0, \underline{\Delta}^T \underline{z} + p + \rho + P \leq 0$ . Then  $r_1^0 = 0, F^0(u|\underline{z}) = I_0(u)$

$$w_i(\underline{z}) = \underline{z}^T D \underline{z} + (\underline{d} + \underline{\xi})^T \underline{z} + \delta + q + Q$$

ii.  $P \geq 0, \underline{\Delta}^T \underline{z} + p + \rho + P \geq 0$ . Then  $r_1^0 = 1, F^0(u|\underline{z}) = I_1(u)$

$$w_i(\underline{z}) = \underline{z}^T D \underline{z} + (\underline{d} + \underline{\xi} + \underline{\Delta})^T \underline{z} + \delta + q + Q + p + \rho + P$$

iii.  $P < 0, 0 \leq -(\underline{\Delta}^T \underline{z} + p + \rho)/2P \leq 1$ . Then

$$r_1^0 = -\frac{(\underline{\Delta}^T \underline{z} + p + \rho)}{2P}, F^0(u|\underline{z}) = I_{r_1^0}(u) \text{ and}$$

$$w_i(\underline{z}) = \underline{z}^T \left( D - \frac{\underline{\Delta} \underline{\Delta}^T}{4P} \right) \underline{z} + \left( \underline{d} - \frac{(p+\rho)\underline{\Delta}}{2P} \right)^T \underline{z} +$$

$$+ \delta + q + Q - \frac{(p+\rho)^2}{4P}$$

(Cont'd)

(5.39)

iv.  $P < 0$ ,  $-\frac{(\underline{\Delta}^T \underline{z} + p + \rho)}{2P} \leq 0$ . Same result as i

v.  $P < 0$ ,  $-\frac{(\underline{\Delta}^T \underline{z} + p + \rho)}{2P} \geq 1$ . Same result as ii.

Case 3. The line  $s_0 = 0$ ,  $s_1 + s_2 = 0$  of  $P_S^*$ . We note that this case applies only to  $s_2 \leq 0$ . It is more complicated than those above because the separating hyperplanes of the two sets, which imply the strategy for Player II, no longer are in one-to-one correspondence with the points of contact. Thus we must examine the slope at the contact point of the boundary of  $S(A(\underline{z}), R, w_i(\underline{z}))$ .

The condition  $s_1 + s_2 = 0$  gives

$$r_1^0 = -(\underline{\xi}^T \underline{z} + q + Q)/\rho \quad (5.40)$$

which must be substituted in the appropriate equation of

$$a. \quad \alpha = \underline{z}^T D\underline{z} + \underline{d}^T \underline{z} + \delta + (\underline{\Delta}^T \underline{z} + p)r_1 + Pr_1^2 \quad (5.41)$$

$$b. \quad \alpha = \underline{z}^T D\underline{z} + \underline{d}^T \underline{z} + \delta + (\underline{\Delta}^T \underline{z} + p + P)r_1$$

provided of course that  $r_1^0 \in [0, 1]$ , a necessary condition for Case 3 to hold all. The following cases may be found.

- i.  $P \geq 0$ . Then  $F^0(u|\underline{z}) = (1 - r_1^0)I_0(u) + r_1^0I_1(u)$ ; that is, the maximizer uses a mixed strategy of points  $u = 0$  and  $u = 1$ , a condition which is clearer if Figure 5-1 is examined and the discussion of Section 4.6 is remembered. It can be seen that

$$w_1(\underline{z}) = \underline{z}^T \left( D - \frac{\Delta \xi^T}{\rho} \right) \underline{z} + \left( \underline{d} - \frac{(p+P)\xi}{\rho} - \frac{(q+Q)\Delta}{\rho} \right)^T \underline{z} + \delta - \frac{(p+P)(q+Q)}{\rho}$$

This results simply by substituting (5.40) into (5.41b).

Using (5.32) we find that, by eliminating  $r_1$  with  $r_2 = r_1$ ,

$$\frac{\partial s_0}{\partial s_1} = \frac{\Delta^T \underline{z} + p + P}{\rho} \quad (5.42)$$

Equation (5.42) along with the condition  $s_0 = 0$ ,  $s_1 + s_2 = 0$  can be used to show that  $\underline{s}^0 = \left( 1 - \frac{\partial s_0}{\partial s_1} - \frac{\partial s_0}{\partial s_1} \right)^T$  or that  $G^0(v|\underline{z}) = \left( 1 + \frac{\partial s_0}{\partial s_1} \right) I_0(v) - \frac{\partial s_0}{\partial s_1} I_1(v)$ . If  $r_1^0 = 0$  or  $r_1^0 = 1$ , this result may not give a separating hyperplane; one of the extremal strategies  $I_0(v)$  or  $I_1(v)$  is then optimum, although not necessarily uniquely so.

- ii.  $P \leq 0$ . In this case  $F^0(u|\underline{z}) = I_{r_1^0}(u)$  where  $r_1^0$  is given by (5.40). Substituting into (5.41a) yields

$$w_1(\underline{z}) = \underline{z}^T \left( D - \frac{\Delta \xi^T}{\rho} + \frac{P}{\rho^2} \xi \xi^T \right) \underline{z} + \left( \underline{d} - \frac{P}{\rho} \xi - \frac{(q+Q)}{\rho} \Delta + \frac{2(q+Q)P}{\rho^2} \xi \right)^T \underline{z} + \delta - \frac{P}{\rho} (q+Q) + \frac{P}{\rho^2} (q+Q)^2 \quad (5.43)$$

Since  $s_2 = Q$  at the point of contact,  $s_1 = -Q$ . Therefore we have

$$\frac{\partial s_0}{\partial s_1} = \frac{(\rho \underline{\Delta} - 2P\underline{\xi})^T}{\rho^2} \underline{z} + \frac{\rho p - 2PQ - 2Pq}{\rho^2} \quad (5.44)$$

and  $G^0(v|\underline{z})$  depends upon  $\frac{\partial s_0}{\partial s_1}$  as in part i of this case.

Case 4. The surface  $4s_0s_2 - s_1^2 = 0$  of  $P_{\xi}^*$ . In this region we concern ourselves with tangency of  $S(A(\underline{z}), R, \alpha)$  and  $P_{\xi}^*$ . Note that at a point of  $P_{\xi}^*$  in this region there can be only one support hyperplane, namely, that corresponding to

$$\underline{s}^0 = \begin{bmatrix} 1 \\ (-\frac{\partial s_0}{\partial s_1}) \\ (-\frac{\partial s_0}{\partial s_1})^2 \end{bmatrix} \quad (5.45)$$

where

$$\frac{\partial s_0}{\partial s_1} = \frac{+s_1}{2s_2} \quad (5.46)$$

Using (5.32) we find that

$$\frac{\partial s_0}{\partial s_1} = \frac{\underline{\xi}^T \underline{z} + q + \rho r_1^0}{Q} \quad (5.47)$$

where  $r_1^0$  is the first moment of the maximizer's optimal strategy.

Substituting (5.33) and (5.32) into the equation (5.34) for the surface, we find

$$\begin{aligned}
\text{a. } \alpha &= \underline{z}^T D \underline{z} + \underline{d}^T \underline{z} + \delta + (\underline{\Delta}^T \underline{z} + p)r_1 + Pr_1^2 - \frac{1}{4Q}(\underline{z}^T \underline{\xi} \underline{\xi}^T \underline{z} \\
&\quad + 2\rho r_1 \underline{\xi}^T \underline{z} + 2q \underline{\xi}^T \underline{z} + q^2 + 2\rho q r_1 + \rho^2 r_1^2) \\
&= \underline{z}^T (D - \frac{\underline{\xi} \underline{\xi}^T}{4Q}) \underline{z} + (\underline{d} - \frac{q}{2Q} \underline{\xi})^T \underline{z} + (\delta - \frac{q^2}{4Q}) \\
&\quad + (\underline{\Delta} - \frac{\rho}{2Q} \underline{\xi})^T \underline{z} r_1 + (p - \frac{\rho q}{2Q}) r_1 + (P - \frac{\rho^2}{4Q}) r_1^2
\end{aligned} \tag{5.48}$$

$$\begin{aligned}
\text{b. } \alpha &= \underline{z}^T (D - \frac{\underline{\xi} \underline{\xi}^T}{4Q}) \underline{z} + (\underline{d} - \frac{q}{2Q} \underline{\xi})^T \underline{z} + (\delta - \frac{q^2}{4Q}) \\
&\quad + (\underline{\Delta}^T \underline{z} - \frac{\rho}{2Q} \underline{\xi}^T \underline{z} + p - \frac{\rho q}{2Q} + P)r_1 - \frac{\rho^2}{4Q} r_1^2
\end{aligned}$$

Implicit in Case a. is that  $P < 0$ , while the contrary holds in Case b. Cases a. and b. are so similar in analysis that we shall treat them together, writing

(5.49)

$$\alpha = \underline{z}^T (D - \frac{\underline{\xi} \underline{\xi}^T}{4Q}) \underline{z} + (\underline{d} - \frac{q}{4Q} \underline{\xi})^T \underline{z} + (\delta - \frac{q^2}{4Q}) + x r_1 + y r_1^2$$

where the definitions of  $x$  and  $y$  should be obvious. Then we have the following situations.

- i. If  $y \geq 0$ ,  $x + y \leq 0$ . Then  $r_1^0 = 0$ ,  $F^0(u|\underline{z}) = I_0(u)$ ,  $G^0(v|\underline{z}) = I_{s^0}(v)$  where  $s^0 = -(\underline{\xi}^T \underline{z} + q)/Q$ , and  $w_1(\underline{z}) = \underline{z}^T (D - \frac{\underline{\xi} \underline{\xi}^T}{4Q}) \underline{z} + (\underline{d} - \frac{q}{4Q} \underline{\xi})^T \underline{z} + (\delta - \frac{q^2}{4Q})$
- ii. If  $y \geq 0$ ,  $x + y > 0$ . Then  $r_1^0 = 1$ ,  $F^0(u|\underline{z}) = I_1(u)$ ,  $G^0(v|\underline{z}) = I_{s^0}(v)$  where  $s^0 = -(\underline{\xi}^T \underline{z} + q + \rho)/Q$  and

(Cont'd)

$$w_i(\underline{z}) = \underline{z}^T \left( D - \frac{\xi \xi^T}{4Q} \right) \underline{z} + \left( \underline{d} - \frac{q}{4Q} \xi \right)^T \underline{z} + \left( \delta - \frac{q^2}{4Q} \right) + x + y$$

iii. If  $y < 0$ , and  $0 \leq \frac{-x}{2y} \leq 1$ . Then  $r_1^0 = \frac{-x}{2y}$ ;

$F^0(u|\underline{z}) = I_0(u)$  if  $P < 0$  (i.e., Case a. is pure strategy) and  $F^0(u|\underline{z}) = (1 - r_1^0) I_0(u) + r_1^0 I_1(u)$  if  $P \geq 0$ .

The minimizer uses the pure strategy  $G^0(v|\underline{z}) = I_{s_0^0}(v)$

where

$$s_0^0 = - \frac{(\xi^T \underline{z} + q - \frac{qx}{2y})}{Q},$$

and the value is

$$w_i(\underline{z}) = \underline{z}^T \left( D - \frac{\xi \xi^T}{4Q} \right) \underline{z} + \left( \underline{d} - \frac{q}{4Q} \xi \right)^T \underline{z} + \left( \delta - \frac{q^2}{4Q} \right) - \frac{x^2}{4y}$$

iv. If  $y < 0$  and  $\frac{-x}{2y} < 0$ , then  $r_1^0 = 0$  (because  $r_1^0 \in [0, 1]$ ) and the optimal strategies and value of  $i$  occur.

v. If  $y < 0$  and  $\frac{-x}{2y} > 1$ , then  $r_1^0 = 1$  and the results of ii. apply.

We have demonstrated by exhausting the possibilities that  $w_i(\underline{z})$  is piecewise quadratic if  $w_{i+1}(\underline{z})$  is quadratic, and by extension we see that if  $w_{i+1}(\underline{z})$  is piecewise quadratic then  $w_i(\underline{z})$  will

necessarily be so also. This completes our induction step and shows that for the linear quadratic game with scalar controls the principle of optimality and the method of dual cones may be applied to arrive at a solution.

A trio of remarks may be made about the constructions above. First, we have not been particularly concerned about the lack of uniqueness of solutions, a fact that may seem to obscure the semicontinuity of the solutions. Nevertheless, the semicontinuity holds. Second, we note that the optimal first moments are either extremal elements of  $[0, 1]$  or are linearly related to  $z$ . Finally, it can be observed in the solutions that for Player I to have optimal mixed strategies, it is necessary that

$$P = \underline{\alpha}_i^T D_{i+1} \alpha_i + P_i \geq 0 \quad (5.50)$$

For the minimizing player to have such strategies, the condition

$$Q = \underline{\beta}_i^T D_{i+1} \beta_i + Q_i \leq 0 \quad (5.51)$$

must hold. These conditions are of course not sufficient.

## 5.5 A LINEAR QUADRATIC GAME WITH VECTOR CONTROLS

If the controls of Section 5.4 are vectors rather than scalars, then the value function is still piecewise quadratic. This is a fact of fundamental importance, for it is a characterization of the solution for a common class of games. It is proven in this section by a technique which is in the spirit of Section 5.4, but which is of necessity not exhaustive in nature. The approach is to show that

for an arbitrary pair of surfaces, one from  $P_S^*$  and the second from  $S(A(\underline{z}), R, \alpha)$ , the value function implied by use of Theorem 4.5 is quadratic in  $\underline{z}$ . Piecewise quadraticity follows immediately. Because of the nature of this proof, it is concerned only with the form of the solution, although the techniques might be used to find the exact solution if that were desired.

The problem of concern to us has dynamics given by

$$\underline{z}(i+1) = T_i \underline{z}(i) + \alpha_i \underline{u}(i) + \beta_i \underline{v}(i) + \underline{\gamma}(i) \quad (5.52)$$

and payoff function, for the truncated game starting at stage  $j$ ,

$$\begin{aligned} J_j = & \underline{z}^T(N+1) \underline{e}_{N+1} \underline{z}(N+1) + \underline{e}_{N+1}^T \underline{z}(N+1) + \epsilon_{N+1} \\ & + \sum_{i=j}^N (\underline{z}^T(i) \underline{c}_i \underline{z}(i) + \underline{e}_i^T \underline{z}(i) + \underline{z}^T(i) \underline{A}_i \underline{u}(i) \\ & + \underline{z}^T \underline{\xi}_i \underline{v}(i) + \underline{u}^T(i) \underline{P}_i \underline{u}(i) + \underline{p}_i^T \underline{u}(i) + \underline{v}^T(i) \underline{Q}_i \underline{v}(i) \\ & + \underline{q}_i^T \underline{v}(i) + \underline{u}^T(i) \underline{\rho}_i \underline{v}(i)) \end{aligned} \quad (5.53)$$

where  $\underline{z}$  is an  $l$ -vector,  $\underline{u}$  is an  $m$ -vector to be chosen from an  $m$ -dimensional unit hypercube  $U$ ,  $\underline{v}$  is an  $n$ -vector to be chosen from an  $n$ -dimensional unit hypercube  $V$ , and  $T_i, \alpha_i, \beta_i, \underline{c}_i, \underline{A}_i, \underline{\xi}_i, \underline{P}_i, \underline{Q}_i, \underline{\rho}_i$  are known matrices of suitable size,  $\underline{\gamma}_i, \underline{e}_i, \underline{p}_i, \underline{q}_i$  are known vectors, and  $\epsilon_{N+1}$  is a scalar constant. We are concerned with proving that the value  $w_j(\underline{z})$  is piecewise quadratic in  $\underline{z}$ , where

$$w_j(\underline{z}) = (\underline{u}(i), \underline{v}(i), i=j, \dots, N) \text{val} [J_j] \quad (5.54)$$

Note that  $w_{N+1}(\underline{z})$  is indeed of the required form. For our induction hypothesis, we assume that  $w_{i+1}(\underline{z})$  is piecewise quadratic, so that

$$w_{i+1}(\underline{z}) = \underline{z}^T D_{i+1} \underline{z} + \underline{d}_{i+1}^T \underline{z} + \delta_{i+1} \quad (5.55)$$

for some  $D_{i+1}$ ,  $\underline{d}_{i+1}$ ,  $\delta_{i+1}$  in a given region of interest, and prove that  $w_i(\underline{z})$  is also of this form. Using the principle of optimality

$$\begin{aligned} w_i(\underline{z}) = & \text{val}_{(\underline{u}, \underline{v})} [ \underline{z}^T \underline{c}_i \underline{z} + \underline{e}_i^T \underline{z} + \underline{z}^T \underline{\Delta}_i \underline{u} + \underline{z}^T \underline{\xi}_i \underline{v} + \underline{u}^T \underline{P}_i \underline{u} \\ & + \underline{p}^T \underline{u} + \underline{v}^T \underline{Q}_i \underline{v} + \underline{q}^T \underline{v} + \underline{u}^T \underline{\rho} \underline{v} + w_{i+1}(\underline{T}_i \underline{z} + \underline{\alpha}_i \underline{u} \\ & + \underline{\beta}_i \underline{v} + \underline{\gamma}_i) ] \end{aligned} \quad (5.56)$$

which after substitutions and definitions in a manner similar to that of the previous section gives the form.

$$\begin{aligned} w_i(\underline{z}) = & \text{val}_{(\underline{u}, \underline{v})} [ \underline{z}^T \underline{D} \underline{z} + \underline{d}^T \underline{z} + \underline{u}^T \underline{P} \underline{u} + \underline{v}^T \underline{Q} \underline{v} + \underline{p}^T \underline{u} + \underline{q}^T \underline{v} \\ & + \underline{u}^T \underline{\rho} \underline{v} + \delta + \underline{z}^T \underline{\Delta} \underline{u} + \underline{z}^T \underline{\xi} \underline{v} ] \end{aligned} \quad (5.57)$$

At this point we define functions  $\underline{r}(\underline{u})$  and  $\underline{s}(\underline{v})$  and a matrix  $A(\underline{z})$  so that (5.57) may be put in standard form. For clarity of presentation we utilize notation which is somewhat more appropriate to matrices than to vectors in that double subscripting of vectors is used. To be consistent with our previous work, however, we continue to work with vectors and matrices rather than create

awkward definitions for some of the sets involved. Thus we define

$$\begin{aligned}
 r_{ij}(\underline{u}) &\triangleq u_i u_j & i=1, 2, \dots, m & & u_0 &\equiv 1 \\
 & & j=0, i, i+1, \dots, m & & & \\
 s_{ij}(\underline{v}) &\triangleq v_i v_j & i=1, 2, \dots, n & & v_0 &\equiv 1 \\
 & & j=0, i, i+1, \dots, n & & &
 \end{aligned}
 \tag{5.58}$$

and we define  $\underline{r}$  and  $\underline{s}$  as

$$\underline{r} \triangleq \begin{bmatrix} r_{00} \\ r_{10} \\ r_{20} \\ \vdots \\ r_{m0} \\ r_{11} \\ r_{22} \\ \vdots \\ r_{mm} \\ r_{12} \\ r_{13} \\ \vdots \\ r_{1m} \\ r_{23} \\ r_{24} \\ \vdots \\ r_{2m} \\ \vdots \\ r_{m-1, m} \end{bmatrix} \qquad \underline{s} \triangleq \begin{bmatrix} s_{00} \\ s_{10} \\ s_{20} \\ \vdots \\ s_{n0} \\ s_{11} \\ s_{22} \\ \vdots \\ s_{nn} \\ s_{12} \\ s_{13} \\ \vdots \\ s_{1n} \\ s_{23} \\ s_{24} \\ \vdots \\ s_{2n} \\ \vdots \\ s_{n-1, n} \end{bmatrix}
 \tag{5.59}$$

The ordering of the components of  $\underline{r}$  and  $\underline{s}$  will not generally be of significance to us. With these definitions, (5.57) may be rewritten as (5.60)

$$w_i(\underline{z}) = \max_{\underline{r} \in R} \min_{\underline{s} \in S} \underline{r}^T \begin{bmatrix} \underline{z}^T D \underline{z} + \underline{d}^T \underline{z} + \delta & \underline{z}^T \underline{\xi} + \underline{q}^T & [Q_{11} \ Q_{22} \ \dots \\ & & Q_{n-1, n-1} + Q_{n, n-1}] \\ \Delta^T \underline{z} + \underline{p} & \rho & 0 \\ \begin{bmatrix} P_{11} \\ P_{22} \\ \vdots \\ P_{mm} \\ P_{12} + P_{21} \\ P_{13} + P_{31} \\ \vdots \\ P_{m-1, m} + P_{m, m-1} \end{bmatrix} & 0 & 0 \end{bmatrix} \underline{s}$$

In this equation, we have used as usual the definitions

$$\underline{r} \triangleq \int_U \underline{r}(\underline{u}) dF(\underline{u}),$$

$$\underline{s} \triangleq \int_V \underline{s}(\underline{v}) dG(\underline{v}),$$

$R$  is the convex hull of  $C_R$ ,  $S$  is the convex hull of  $C_S$ , and so on.

The proof proceeds in three major steps. First it is argued by using our knowledge of simple cases and of the nature of a solution

to the problem that the boundaries of R and S must have a certain form. Then we show that the boundaries of  $P_S^*$  must have certain properties. Finally, using this knowledge of R and  $P_S^*$ , the form of  $w_i(z)$  is discussed.

In developing a structure for the boundary of R, we shall exploit the fact that the competitive element of the game appears only through the  $\rho$  matrix in the form of terms  $\rho_{ij} u_i v_j$ . Thus only the first moments  $r_{i0}$ ,  $i=1, 2, \dots, n$  must be chosen with the opponent in mind. The terms  $r_{ij}$ ,  $j \neq 0$ , may be chosen to optimize the payoff, consistent only with the constraints imposed by the value of  $r_{i0}$ . Since we know from the scalar control case (Figure 5-1) that  $r_{ii} \geq r_{i0}^2$  and  $r_{ii} \leq r_{i0}$  are required for any realizable distribution, and since  $r_{ii}$  must be chosen to have minimum or maximum value depending upon the algebraic sign of  $P_{ii}$ , it follows that the boundary regions of R have  $r_{ii}$  related to  $r_{i0}$  by

$$\begin{aligned} \text{a. } \min r_{ii} &= r_{i0}^2 \\ \text{b. } \max r_{ii} &= r_{i0} \end{aligned} \tag{5.61}$$

We may argue in a similar manner concerning the cross-coupling moments  $r_{ij}$ ,  $j, i \neq 0, j \neq i$ . Two separate possibilities arise in this case. If either  $r_{i0}$  or  $r_{j0}$  is associated with a pure strategy, then  $r_{ij} = r_{i0} r_{j0} = E[u_i u_j]$ . If both control elements are associated with mixed strategies, then using the argument about the possibility of choosing  $r_{ij}$  independently of the competition yields that  $r_{ij}$  should be either minimized or maximized within the limits of the chosen

first moments  $r_{i0}$  and  $r_{j0}$ . Some thought and an examination of Figure 4-1 reveals that the maximum value of  $r_{ij}$  is given by

$$\max r_{ij} = \min [r_{i0}, r_{j0}] \quad (5.62)$$

and the minimum by

$$\min r_{ij} = \max [0, r_{i0} + r_{j0} - 1] \quad (5.63)$$

This can also be shown by considering possible bivariate distributions on  $u_i$  and  $u_j$ .

To find  $P_S^*$ , we exploit Theorem 4.6, which says that the boundaries of  $P_S^*$  may be generated using the pure strategies represented by  $C_S$ . A pure strategy for the minimizer will have  $n_t$  elements, say the first  $0 \leq n_t \leq n$ , chosen from  $(0, 1)$ ,  $n_0$  elements,  $0 \leq n_0 \leq n$ , with value zero, and  $n_1$  elements, say the last  $0 \leq n_1 \leq n$ ,  $n_1 = n - n_t - n_0$ , with value one. Let the region of  $C_S$  with this characteristic be denoted  $C'_S$ , so that

$$C'_S = \{ \underline{x} \mid \underline{x} \in E, \frac{v^2 + v + 2}{2}, \underline{x} = \underline{s}(\underline{t}), \text{ where} \quad (5.64)$$

$$t_i \in (0, 1), i=1, 2, \dots, n_t$$

$$t_i = 0, i=n_t+1, \dots, n_t+n_0$$

$$t_i = 1, i=n_t+n_0+1, \dots, n \}$$

and let  $P_S^{*'} be the dual convex cone generated by  $C'_S$ , i. e.,$

$$P_S^* = \left\{ \underline{s} \mid \underline{s} \in E \frac{v^2 + \nu + 2}{2}, \right. \\ \left. \underline{s}^T \underline{x} \geq 0 \text{ for all } \underline{x} \in C_S' \right\} \quad (5.65)$$

The set  $P_S^*$  consists of the intersection of all halfspaces defined by hyperplanes with representation  $\underline{x} \in C_S'$ , and it is clear that  $P_S^* \subset P_S^*$ . For a given  $\underline{x}(t) \in C_S'$ , this requires that

$$\begin{aligned} \text{a. } & \underline{s}^T \underline{x}(t) = 0 \\ \text{b. } & \left. \frac{\partial}{\partial t_i} [\underline{s}^T \underline{x}(t)] \right|_{\underline{t}} = 0 \quad i=1, 2, \dots, n_t \\ \text{c. } & \left. \frac{\partial}{\partial t_i} [\underline{s}^T \underline{x}(t)] \right|_{\underline{t}} \geq 0 \quad i=n_t+1, 2, \dots, n_t+n_0 \\ \text{d. } & \left. \frac{\partial}{\partial t_i} [\underline{s}^T \underline{x}(t)] \right|_{\underline{t}} \leq 0 \quad i=n_t+n_0+1, \dots, n \end{aligned} \quad (5.66)$$

By using (5.66a) and (5.66b) we may remove the dependence upon  $\underline{t}$  and thus find an equation for the surface of  $P_S^*$  as the relevant components of  $\underline{t}$  vary over  $(0, 1)$ . To do this, we expand (5.66b) to get, using the notational conventions defined previously for  $\underline{s}$ ,

$$s_{i0} + 2t_i s_{ii} + \sum_{j=1}^{i-1} s_{ji} t_j + \sum_{j=i+1}^n s_{ij} t_j = 0 \quad i=1, 2, \dots, n_t \quad (5.67)$$

When the known values  $t_j = 0$  and  $t_k = 1$  are substituted into (5.67), there remain  $n_t$  linear equations in the  $n_t$  unknowns  $t_i$ ,  $i=1, 2, \dots, n_t$ . These may be represented in the form

(5.68)

$$\begin{bmatrix} 2s_{11} & s_{12} & \cdots & s_{1n_t} \\ s_{12} & 2s_{22} & & s_{2n_t} \\ s_{13} & s_{23} & 2s_{33} & s_{3n_t} \\ \vdots & \vdots & \vdots & \vdots \\ s_{1n_t} & s_{2n_t} & & 2s_{n_t n_t} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_{n_t} \end{bmatrix} = - \begin{bmatrix} s_{10} + \sum_{i=n_t+n_0+1}^n s_{1i} \\ \vdots \\ s_{n_t 0} + \sum_{i=n_t+n_0+1}^n s_{n_t i} \end{bmatrix}$$

Suppose (5.68) were solved for the components  $t_i$ ,  $i=1, 2, \dots, n_t$  and the results substituted in (5.66a). In solving for the  $t_i$ , any denominator terms will contain only elements  $s_{ij}$  which corresponded to quadratic elements  $t_i^2$  or  $t_i t_j$  in (5.66a). Furthermore, numerators will contain terms for which  $t_j = 0$  or  $t_j = 1$  or terms which correspond to linear functions of  $t_j$ , that is, elements  $s_{i0}$ . Finally,  $s_{00}$  does not appear in the solutions for the elements  $t_i$ . Thus inserting the expressions for  $t_i$  in (5.66a) and clearing of fractions gives an equation of the form

$$s_{00} h_0(\underline{s}) + \sum_{i=1}^n (s_{i0} h_i(\underline{s})) + \sum_{j=1}^n s_{i0} s_{j0} h_{ij}(\underline{s}) + H(\underline{s}) = 0 \tag{5.69}$$

where the functions of  $\underline{s}$  indicated are functions only of the higher order terms  $s_{ij}$ ,  $i, j \neq 0$ . Many of the functions are in fact zero and are retained only to keep the expression (5.69) simple and symmetrical, since their exact nature is unimportant for our purposes.

Having developed characteristics (5.61), (5.62), and (5.63) of the boundary of  $R$  and characteristics (5.69) of the boundary of  $P_S^*$ , we proceed to examine the nature of  $w_i(\underline{z})$ . In the usual manner we bias the  $(0, 0)$  term of the matrix of (5.60) by subtracting a parameter  $\alpha$  and then forming  $S(A(z), R, \alpha)$ . From (5.60) we see that a particular element  $\underline{r} \in R$  is mapped as follows into  $\underline{s}$ -space.

$$s_{00} = \underline{z}^T D \underline{z} + \underline{d}^T \underline{z} + \delta + (\underline{z}^T \Delta + \underline{p}^T) \begin{bmatrix} r_{10} \\ r_{20} \\ \vdots \\ r_{m0} \end{bmatrix} + \sum_{i=1}^m r_{ii} P_{ii} + \sum_{i=1}^{m-1} \sum_{j=i+1}^m (P_{ij} + F_{ji}) r_{ij} - \alpha \quad (5.70)$$

$$\begin{bmatrix} s_{10} \\ s_{20} \\ \vdots \\ s_{n0} \end{bmatrix} = \xi^T \underline{z} + \underline{q} + \rho^T \begin{bmatrix} r_{10} \\ r_{20} \\ \vdots \\ r_{m0} \end{bmatrix}$$

$$s_{ii} = Q_{ii} \quad i=1, 2, \dots, n$$

$$s_{ij} = Q_{ij} + Q_{ji} \quad i=1, 2, \dots, n-1; j=i+1, i+2, \dots, n$$

These coordinates must lie, for the maximum  $\alpha$ , on the boundary of  $P_S^*$ , and thus must satisfy (5.69). Substituting (5.70) into (5.69), recognizing that  $s_{ij}$  is a constant for  $i, j \neq 0$  and that

$s_{i0}$  is linear in  $\underline{z}$  and in  $\underline{r}_{j0}$ , and using the fact that  $h_0(\underline{s}) \neq 0$  by the nature of  $F^*$ , we can write the  $s_{00}$  point in the hyperplane corresponding to  $s_{ij}, i, j \neq 0$ , of  $S(A(\underline{z}), R, \alpha)$  in the form (for suitable constant matrices and vectors)

$$s_{00} = c_0 + \underline{c}_1^T \underline{\tilde{r}} + \underline{z}^T C_2 \underline{\tilde{r}} + \underline{z}^T C_3 \underline{z} + \underline{\tilde{r}}^T C_4 \underline{\tilde{r}} \quad (5.71)$$

Here we define

$$\underline{\tilde{r}} = \begin{bmatrix} r_{10} \\ r_{20} \\ \cdot \\ \cdot \\ r_{m0} \end{bmatrix} \quad (5.72)$$

It is noteworthy that  $s_{00}$  in (5.71) depends only on the first moments  $r_{i0}$  of the maximizer's strategy. Substituting (5.71) into the first equation of (5.70) and solving for  $\alpha$  yields the form, for suitable matrices and vectors

$$\begin{aligned} \alpha = & \underline{z}^T B_1 \underline{z} + \underline{b}_2^T \underline{z} + b_3 + \underline{b}_4^T \underline{\tilde{r}} - \underline{z}^T B_5 \underline{\tilde{r}} - \underline{\tilde{r}}^T C_4 \underline{\tilde{r}} + \\ & + \sum_{i=1}^m P_{ii} r_{ii} + \sum_{i=1}^{m-1} \sum_{j=i+1}^m (P_{ij} + P_{ji}) r_{ij} \end{aligned} \quad (5.73)$$

It is necessary that  $\underline{r} \in R$  be chosen to maximize  $\alpha$ ; the maximum of  $\alpha$  will be  $w_i(\underline{z})$ .

The structure of the boundary of  $R$  may now be exploited. Parameterize (5.73) by letting  $r_{i0} = t_i, i=1, 2, \dots, m, t_i \in [0, 1]$ .

The boundary region of interest is such that it generates some pure strategies and some mixed strategies for components of  $\underline{u}$ . Without loss of generality, let the first  $m'$  components,  $0 \leq m' \leq m$ , be associated with pure strategies, and let the final  $m-m'$  be mixed. Then (5.61) implies

$$\begin{aligned} r_{ii} &= t_i^2 & i=1, 2, \dots, m' \\ r_{ii} &= t_i & i=m'+1, \dots, m \end{aligned} \quad (5.74)$$

For the  $r_{ij}$ ,  $m \geq j > i > m'$ , for which mixed strategy cross-coupling occurs, we may suppose that the coefficients  $(P_{ij} + P_{ji})$  in (5.73) are such that, using (5.62) and (5.63)

$$\begin{aligned} r_{ij} &= r_{i0} & (i, j) \in K_1 \\ r_{ij} &= r_{j0} & (i, j) \in K_2 \\ r_{ij} &= 0 & (i, j) \in K_3 \\ r_{ij} &= r_{i0} + r_{j0} - 1 & (i, j) \in K_4 \end{aligned} \quad (5.75)$$

where the  $K_i$  are sets of integer pairs, and  $K_1 \cup K_2 \cup K_3 \cup K_4$  is the set of all  $(i, j)$  pairs,  $m \geq j > i > m'$ . Then (5.73) becomes

$$\alpha = \underline{z}^T B_1 \underline{z} + \underline{b}_2^T \underline{z} + b_3 + \underline{b}_4^T \underline{t} - \underline{z}^T B_5 \underline{t} - \underline{t}^T C_4 \underline{t} + \sum_{i=1}^{m'} P_{ii} t_i^2 + \quad (5.76)$$

$$+ \sum_{i=m'+1}^m P_{ii} t_i + \sum_{(i,j) \in K_1} (P_{ij} + P_{ji}) t_i + \sum_{(i,j) \in K_2} (P_{ij} + P_{ji}) t_j +$$

(Cont'd)

$$+ \sum_{(i,j) \in K_4} (P_{ij} + P_{ji})(t_i + t_j - 1) \quad (5.76)$$

The maximization of  $\alpha$  over  $t_i \in [0, 1]$ ,  $i=1, 2, \dots, m$  may now be performed. Some  $t_i$  appear linearly in (5.76) and take on values of either 0 or 1 according to the signs of their coefficients. For these  $t_i$  which appear quadratically, we find the inflection point of (5.76)

$$\frac{\partial \alpha}{\partial t_i} = 0 = (\underline{b}_4)_i - \underline{z}^T (\underline{B}_5)_i - \underline{t}^T ((\underline{C}_4)_i - (\underline{C}_4^T)_i) + \tilde{P} + P'(t_i) \quad (5.77)$$

where the notation  $(\cdot)_i$  indicates  $i^{\text{th}}$  element or column and

$$\tilde{P} = \sum_K (P_{ij} + P_{ji}) \quad K \text{ is set of applicable } (i, j) \quad (5.78)$$

$$P'(t_i) = \begin{cases} P_{ii} & m \geq i > m' \\ 2P_{ii} t_i & 1 \leq i \leq m' \end{cases}$$

Equations (5.78) are purposely left vague, since they depend upon which sets  $K_k$  contain index  $i$ , and in what manner it is contained. This is not important to our argument, since  $\tilde{P}$  is constant in any case. The set of equations (5.77) is linear in  $\underline{z}$  and  $\underline{t}$ , and the coefficients of  $\underline{t}$  are known constants. The equation set may in principle be solved so that  $t_i \in [0, 1]$ , although in practice constraining the values to this bounded set may be a nuisance. A

solution, perhaps not unique, must exist by the nature of the problem, and after all the extremal values of  $t_i$  have been found, there will remain a set equations of the form (5.77) in which some number  $k$  of the components of  $\underline{t}$  are unknown and the same number  $k$  of equations may be solved. It is clear that the unknown components must be linear functions of  $\underline{z}$ , a crucial point.

Therefore the elements  $t_i$ ,  $i=1, 2, \dots, m$  which maximize  $\alpha$  are either zero or one in value or are linear functions of  $\underline{z}$ . Substituting them into (5.76) clearly gives the desired result, i. e.,  $\alpha_{\max} = w_1(\underline{z})$  is a quadratic function of  $\underline{z}$ .

Since both  $S(A(\underline{z}), R, \alpha)$  and  $P_{\xi}^*$  must by their nature have finite numbers of recognizable surfaces, i. e., boundary regions for which a single equation set or parameterization rule may be used to describe the region, the arguments above may be repeated for each pair of surfaces. Therefore  $w_1(\underline{z})$  is piecewise quadratic. We have proven the following theorem.

**Theorem 5.7:** The  $N$ -stage game starting at stage  $i$  with linear dynamics, quadratic payoff function, and controls chosen from unit hypercubes at each stage has a piecewise quadratic value function.

This theorem holds whether open-loop or closed-loop strategies are involved. It is particularly significant for the closed loop case, for it implies that the principle of optimality may be applied to give exact solutions. It is also significant for numerical solutions,

since computation of the value is then reduced to determination of coefficients.

## 5.6 SUMMARY

In this chapter certain multistage games were shown to be reducible to sequences of separable static games in which the state vector is a simple parameter. The continuity characteristics of the optimal solutions were then extensively investigated. Finally, the method of dual cones was applied to linear-quadratic games and it was demonstrated that the value function is not only continuous, but piecewise quadratic.

The implications of these results are obvious: certain dynamic games can be solved. This can be done, at least in principle, for all games (of the class studied here) with open loop strategies and for linear-quadratic games with closed loop strategies; it may also be possible for other games. Furthermore, the continuity properties and the nature of the method of dual cones guarantee that numerical approximation is both straightforward and appropriate. This latter point should prove to be particularly important for applications.

## CHAPTER 6

### EXAMPLES

In this chapter are several examples which illustrate the ideas involved in solving polynomial multistage games using the method of dual cones. The examples are of low dimension so that the geometric interrelationships may be visualized and are motivated by using a multistage formulation even when it is not the multistage character which is of primary interest. The demonstrative value of the models is emphasized rather than the intrinsic value.

#### 6.1 A LINEAR-QUADRATIC SCALAR PROBLEM

The first example is an extremely simple one which we shall examine in detail; its simplicity is such that we may concentrate on our techniques and not be distracted by algebraic detail.

Let  $z$  be a scalar state variable and let  $u' \in [-\frac{1}{2}, \frac{1}{2}]$ ,  $v' \in [-\frac{1}{2}, \frac{1}{2}]$  be scalar controls for a system with dynamics

$$z(i+1) = z(i) + u'(i) + v'(i) \quad (6.1)$$

Suppose that an  $N$ -stage game with final value payoff

$$J = z^2(N+1) \quad (6.2)$$

is to be played using this system, with player I choosing  $u'(i)$  and maximizing and player II choosing  $v'(i)$  and minimizing, where  $i=1, 2, \dots, N$ . Let us agree, since the parameters are scalars, to use subscripts to indicate the stage index,  $z_i = z(i)$ , etc., and

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let us transform the controls using  $u_i = u'(i) + \frac{1}{2}$ ,  $v_i = v'(i) + \frac{1}{2}$  so that the dynamics (6.1) become

$$z_{i+1} = z_i - 1 + u_i + v_i \quad (6.3)$$

where  $u_i \in [0, 1]$ ,  $v_i \in [0, 1]$  as required by our paradigm.

The solution to this problem appears intuitively obvious except near the origin  $z = 0$ : the maximizer will choose his control to get as far from the origin,  $z = 0$ , as possible and the minimizer will attempt to cause  $z_{N+1}$  to be near the origin. Thus for  $z_i \gg 0$ , for example,  $u_i = \frac{1}{2}$ ,  $v_i = -\frac{1}{2}$  is obvious, so that  $z_{i+1} = z_i$  and  $z_{N+1} = z_i$ . For  $z_i \approx 0$ , however, intuition is not so helpful; e.g., if  $z_N = 0$ , then

$$\min_{v_N} \max_{u_N} z_{N+1}^2 = 0 \quad (6.4)$$

$$\max_{u_N} \min_{v_N} z_{N+1}^2 = 1$$

and the need for a mixed strategy for one or both players is apparent. We shall find those mixed strategies and also verify the intuitive pure strategies.

Let us first solve the single-stage, or one-stage-to-go, problem. For ease of notation, define  $u = u_N$ ,  $v = v_N$ ,  $z = z_N - 1$ , so that

$$z_{N+1} = z + u + v \quad (6.5)$$

and

$$J = (z + u + v)^2 = J(z, u, v) \quad (6.6)$$

We seek cumulative distribution functions  $F^0(u|z)$  and  $G^0(v|z)$  such that

$$\begin{aligned} w(z) &= \min_{G(v)} \iint_{V \times U} J(z, u, v) dF^0(u|z) dG(v) \\ &= \max_{F(u)} \iint_{U \times V} J(z, u, v) dG^0(v|z) dF(u) \end{aligned} \quad (6.7)$$

Expanding  $J$  and writing it in matrix form yields

$$w(z) = \min_{G(v|z)} \max_{F(u|z)} E \left\{ \begin{bmatrix} 1 & u & u^2 \end{bmatrix} \begin{bmatrix} z^2 & 2z & 1 \\ 2z & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} \right\} \quad (6.8)$$

By subtracting  $w(z)$  from both sides and defining

$$r_i = E[u^i] = \int_0^1 u^i dF(u|z) \quad i=0, 1, 2 \quad (6.9)$$

$$s_j = E[v^j] = \int_0^1 v^j dG(v|z) \quad j=0, 1, 2$$

we may write (6.8) as

$$0 = \min_{\underline{s} \in S} \max_{\underline{r} \in R} [1 \quad r_1 \quad r_2] \begin{bmatrix} z^2 - w(z) & 2z & 1 \\ 2z & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s_1 \\ s_2 \end{bmatrix} \quad (6.10)$$

where  $S$  and  $R$  are the sets of admissible moment vectors  $[s_0, s_1, s_2]^T$  and  $[r_0, r_1, r_2]^T$ , respectively, and  $s_0 = r_0 = 1$ .

The set  $C_R$  is given parametrically by  $C_R = \{\underline{x} \mid r_0 = 1, r_1 = t, r_2 = t^2, t \in [0, 1]\}$ , and  $R$  is the convex hull of this set. The significant cross-sections  $\hat{C}_R$  and  $\hat{R}$  are shown in Figure 6-1. We see that  $R = \{\underline{x} \mid r_0 = 1, r_1^2 \leq r_2, r_1 \geq r_2, r_1 \in [0, 1]\}$ . The sets  $C_S$  and  $S$  are identical in form to  $C_R$  and  $R$ .

The cone  $P_S$  is easily constructed using the cross-section  $S$ , i. e.,  $P_S = \{\underline{s} \mid \underline{s} = \lambda \underline{s}' \text{ for some } \lambda \geq 0 \text{ and } \underline{s}' \in S\}$ . This set is drawn in Figure 6-2.

The dual cone  $P_S^*$  is slightly more difficult to visualize. By definition

$$P_S^* = \{\underline{s} \mid \underline{s}^T \underline{x} \geq 0, \forall \underline{x} \in P_S\} \quad (6.11)$$

Let us use one illuminating method of construction. Pick a particular point  $\underline{x}_0 \in P_S$  and consider the set  $P_S^*(\underline{x}_0)$

$$P_S^*(\underline{x}_0) = \{\underline{s} \mid \underline{s}^T \underline{x}_0 \geq 0\} \quad (6.12)$$

This will be a half-space in  $E^3$  with boundary points  $\underline{s}^0$  such that  $\underline{s}^{0T} \underline{x}_0 = 0$  (Figure 6-3). The region in the direction of positive  $s_0$

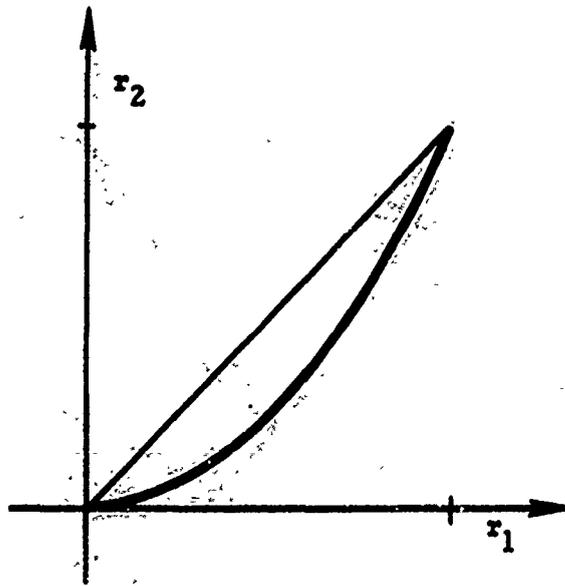


Figure 6-1. The set  $\hat{C}_R$  and its convex hull  $\hat{R}$ .

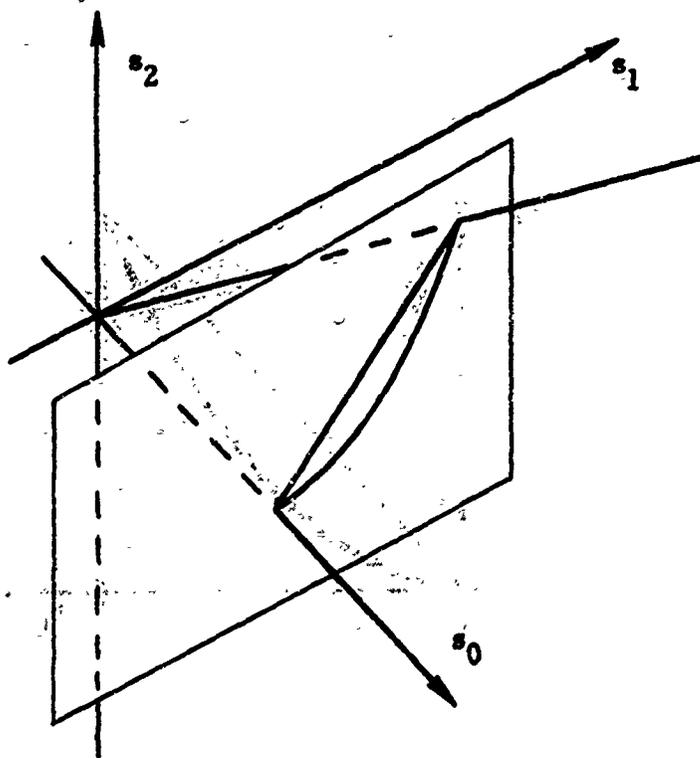


Figure 6-2. The cone  $P_S$ , showing the cross-section  $S$ .

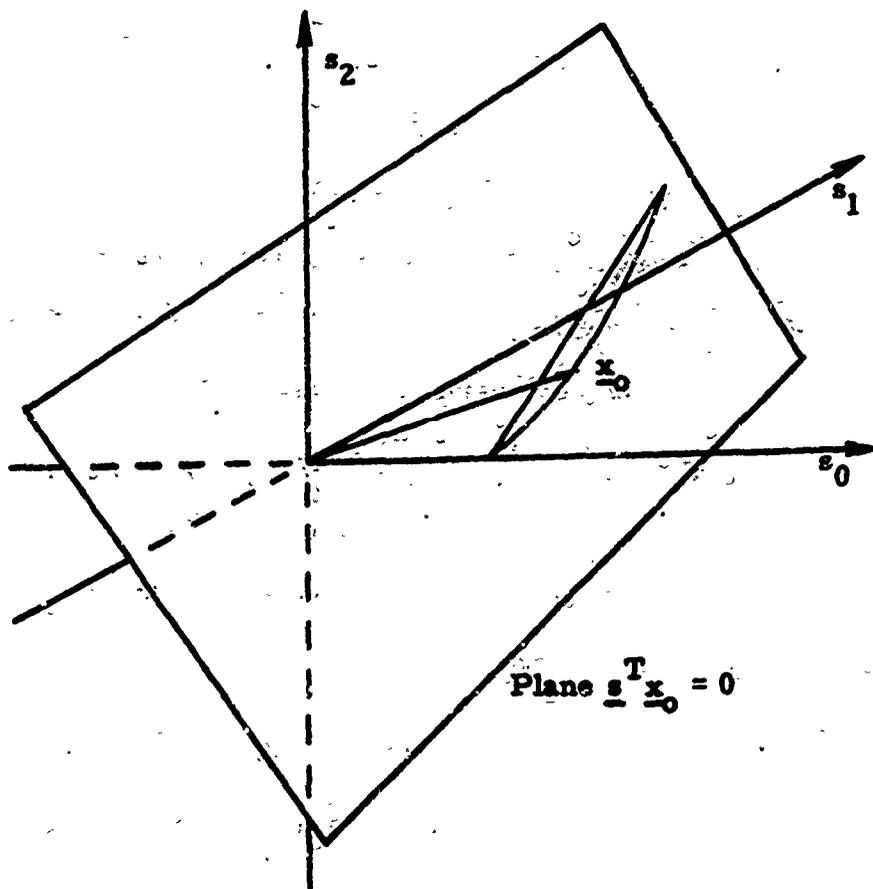


Figure 6-3. A representative half-space containing  $P_S^*$ .

belongs to  $P_S^*(\underline{x}_0)$ . For two points  $\underline{x}_0$  and  $\underline{x}_1$  in  $P_S$ , we see that only points  $\underline{s}$  belonging to both half spaces can belong to  $P_S^*$ ; i. e.,  $\underline{s} \in P_S^*$  implies  $\underline{s} \in P_S^*(\underline{x}_0) \cap P_S^*(\underline{x}_1)$ . In fact  $\underline{s} \in P_S^*$  implies that  $\underline{s} \in P_S^*(\underline{x}_0) \cap P_S^*(\underline{x}_1) \cap \dots \cap P_S^*(\underline{x}_i) \cap \dots$  for all  $\underline{x}_i \in P_S$ . Therefore a boundary point of  $P_S^*$  must belong to  $P_S^*(\underline{x})$  for all  $\underline{x} \in P_S$  and must be a boundary point of  $P_S^*(\underline{x})$  for at least one  $\underline{x} \in P_S$ .

From Theorem 4.6, we know that boundary points of  $P_S^*$  other than the origin can only be generated by points  $\underline{s}$  of  $P_S$  which for some  $\lambda > 0$  have the property  $\lambda \underline{s} \in C_S$ . Hence the construction of the boundary requires consideration only of points  $\underline{s}$  from the set

$$\{\underline{s} \mid \underline{s}^T \underline{x} = 0 \text{ for some } \underline{x} \in C_S, \underline{s}^T \underline{y} \geq 0 \text{ for all } \underline{y} \in C_S\} \quad (6.13)$$

In this example, these comments allow us to restrict our attention to points  $\underline{s}$  which satisfy

$$\begin{aligned} s_0 + s_1 t + s_2 t^2 &= 0 \text{ for some } t \in [0, 1], \\ s_0 + s_1 t' + s_2 t'^2 &\geq 0 \text{ for all } t' \in [0, 1]. \end{aligned} \quad (6.14)$$

If  $t \in (0, 1)$ , then for suitable  $\delta$ ,  $t + \delta \in [0, 1]$ , and (6.14) is equivalent to

$$s_0 + s_1 t + s_2 t^2 = 0 \quad t \in (0, 1)$$

(6.15)

and

$$s_0 + s_1(t + \delta) + s_2(t + \delta)^2 \geq 0 \quad t + \delta \in [0, 1]$$

This implies that

$$\begin{aligned} s_0 + s_1 t + s_2 t^2 &= 0 \\ s_1 \delta + s_2(2t\delta + \delta^2) &\geq 0 \end{aligned} \quad (6.16)$$

Since  $\delta$  may be either positive or negative

$$\begin{aligned} s_0 + s_1 t + s_2 t^2 &= 0, \\ t &\in (0, 1), \\ s_1 + 2t s_2 &= 0, \end{aligned} \quad (6.17)$$

from which  $t$  may be eliminated to give

$$s_0 - \frac{1}{s_2} \left( \frac{-s_1}{2} \right)^2 = 0, \quad -1 < \frac{s_1}{2s_2} < 0 \quad (6.18a)$$

The end points  $t = 0$  and  $t = 1$  yield

$$s_0 = 0 \quad (6.18b)$$

$$s_0 + s_1 + s_2 = 0 \quad (6.18c)$$

as other boundary surfaces. Combining (6.18a)-(6.18c) yields the boundaries of  $P_S^*$  (Figure 6-4). These are more easily visualized if the pair of cross-sections in Figure 6-5 are considered.

With  $R$  and  $P_S^*$  known, we are ready to proceed with the problem solution. Let us use the matrix of (6.10) to map  $R$  into  $S$ -space; i. e., define

$$\begin{aligned} S(A(z), R, f) &= \{ \underline{s} \mid \exists \underline{r} \in R \ni s_0 = z^2 - f + 2zr_1 + r_2, \\ & \quad s_1 = 2z + 2r_1, \quad s_2 = 1 \} \end{aligned} \quad (6.19)$$

For convenience, let us denote  $S(A(z), R, f)$  by  $S(z, f)$ . Then if  $f = w(z)$ ,  $S(z, f)$  intersects  $P_S^*$  only at boundary points. We see that for all  $f$  and  $z$ ,  $\underline{s} \in S(z, f)$  implies  $s_2 = 1$ , so that the intersection of the sets

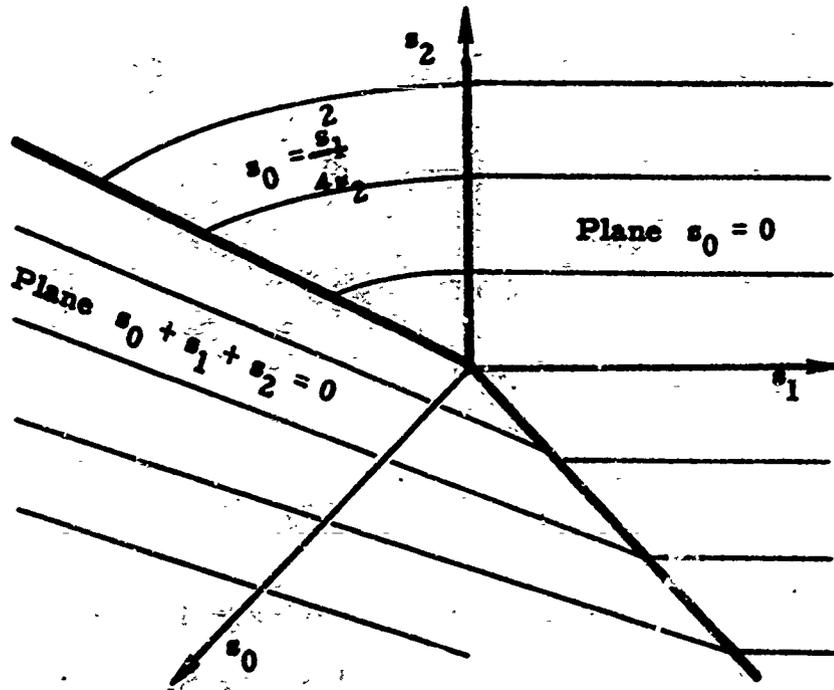
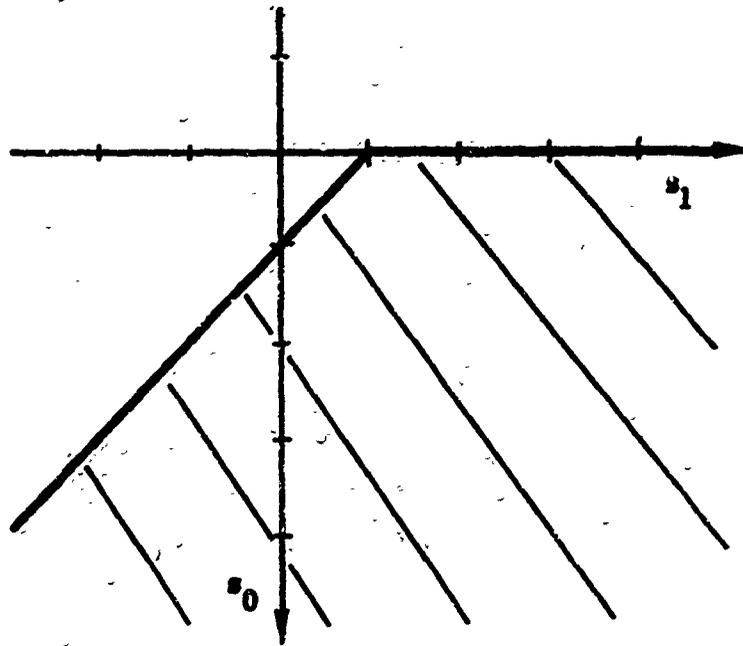
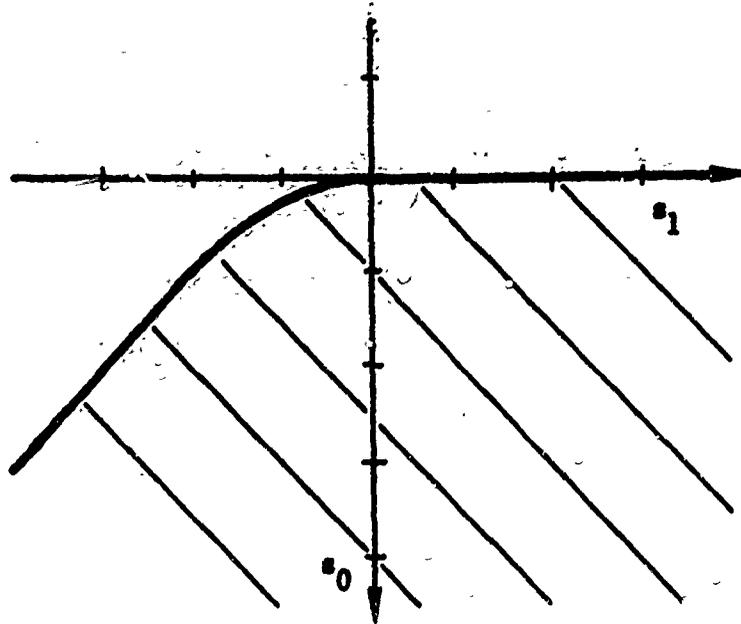


Figure 6-4. Boundaries of the dual cone  $P_S^*$ .



(a)  $s_2 = -1$



(b)  $s_2 = +1$

Figure 6-5. Cross-sections of  $P_S^*$  for Example 1.

must occur for this value of  $s_2$ , and we need only consider the  $s_2 = 1$  cross-section of  $P_S^*$ . This cross-section is given in Figure 6-5(b). Let  $S'(z, f)$  be the projection of  $S(A(z), R, f)$  on the  $s_2 = 1$  plane.

Let us now consider sample values of  $z$  and  $f$  and perform the mapping of (6.19).

$$\begin{aligned} S'(1, 0) &= \{s_0, s_1 \mid \exists r \in R \ni s_0 = 1 + 2r_1 + r_2, s_1 = 2 + 2r_1\} \\ S'(1, 4) &= \{s_0, s_1 \mid \exists r \in R \ni s_0 = -3 + 2r_1 + r_2, s_1 = 2 + 2r_1\} \quad (6.20) \\ S'(1, 2) &= \{s_0, s_1 \mid \exists r \in R \ni s_0 = -1 + 2r_1 + r_2, s_1 = 2 + 2r_1\} \end{aligned}$$

These sets are shown in Figure 6-6. Performing the mapping is aided considerably by the fact that, for given  $z$ , it is a linear mapping. Thus the straight line segment  $r_1 = r_2$  maps into a straight line segment  $s_0 = \frac{3}{2} s_1 - f - 2$ , and the segment of  $r_2 = r_1^2$  maps into a segment of  $s_0 = \frac{1}{4} s_1^2 - f$ .

Examination of Figure 6-6 reveals forcefully the effect of  $f$  in causing the translation of  $S'(z, f)$  parallel to the  $s_0$ -axis. Furthermore, it is obvious that  $w(1)$  is the maximum value of  $f$  for which  $S(1, f) \cap P_S^* \neq \emptyset$ , or alternatively the minimum  $f$  for which a separating plane for  $S(1, f)$  and  $P_S^*$  exists. Since  $f = 4$  has the desired qualities,  $w(1) = 4$ . This occurs for  $r_1 = r_2 = 1$ , so that the pure strategy  $F^0(u) = I_1(u)$  suffices for the maximizer. The separating hyperplane is  $s_0 = 0$ , implying that the pure strategy  $G^0(v) = I_0(v)$  is used by the minimizer. (As usual the function  $I_x(y) = 1$  for  $y \geq x$ ,  $I_x(y) = 0$ ,  $y < x$ , is used.)

Before evaluating  $w(z)$  in general, let us examine two more

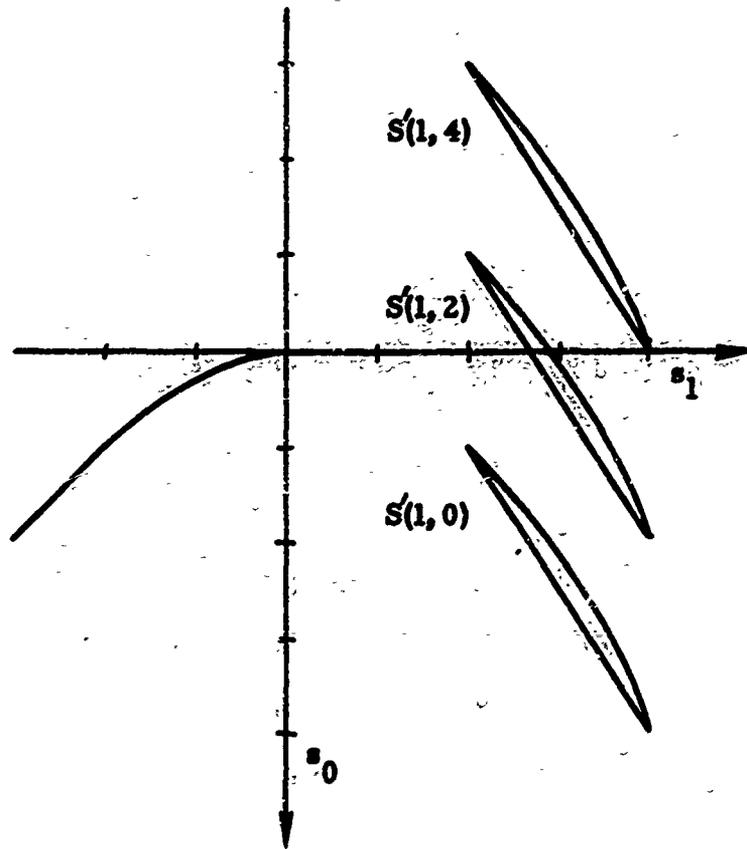


Figure 6-6. Mappings of R into  $\underline{S}$ -space for  $z = 1$ .

sample values of  $z$ .

$$S'(-3, 0) = \{s_0, s_1 \mid \exists \underline{r} \in R \ni s_0 = 9 - 6r_1 + r_2, s_1 = -6 + 2r_1\}$$

$$S'(-3, 4) = \{s_0, s_1 \mid \exists \underline{r} \in R \ni s_0 = 5 - 6r_1 + r_2, s_1 = -6 + 2r_1\}$$

$$S'(-3, 6) = \{s_0, s_1 \mid \exists \underline{r} \in R \ni s_0 = 3 - 6r_1 + r_2, s_1 = -6 + 2r_1\}$$

(6.21)

$$S'(-1, -1) = \{s_0, s_1 \mid \exists \underline{r} \in R \ni s_0 = 2 - 2r_1 + r_2, s_1 = -2 + 2r_1\}$$

$$S'(-1, \frac{1}{4}) = \{s_0, s_1 \mid \exists \underline{r} \in R \ni s_0 = \frac{3}{4} - 2r_1 + r_2, s_1 = -2 + 2r_1\}$$

$$S'(-1, 2) = \{s_0, s_1 \mid \exists \underline{r} \in R \ni s_0 = -1 - 2r_1 + r_2, s_1 = -2 + 2r_1\}$$

These sets are sketched in Figure 6-7. Looking first at the sets  $S(-3, f)$ , we see that  $S(-3, 6)$  does not intersect  $P_S^*$ , that  $S(-3, 0)$  lies entirely within  $P_S^*$  and thus does not have a hyperplane separating it from  $P_S^*$ , and that  $S(-3, 4)$  appears to both intersect and share the separating hyperplane  $s_0 + s_1 = -1$ . Thus it appears that  $w(-3) = 4$ , and  $G^0(v) = I_1(v)$ . Furthermore, the intersection point  $s_0 = 5$ ,  $s_1 = -6$  corresponds to  $r_1 = r_2 = 0$  in  $R$  for  $f = 4$ , and thus  $F^0(u) = I_0(u)$ .

For the sets  $S(-1, f)$ , it appears graphically that  $w(-1) = \frac{1}{4}$ , that the separating plane is  $2s_0 - s_1 = -\frac{1}{4}$ , and that for the point of contact  $s_0 = \frac{1}{4}$ ,  $s_1 = -1$ , the corresponding  $\underline{r} \in R$  is  $r_1 = r_2 = \frac{1}{2}$ . Therefore optimal strategies are  $G^C(v) = I_1(v)$  and  $F^C(u) = \frac{1}{2}I_0(u) + \frac{1}{2}I_1(u)$ , where the latter indicates a 50-50 mix of  $u = 0$  and  $u = 1$  for the maximizer. These values will be verified algebraically below.

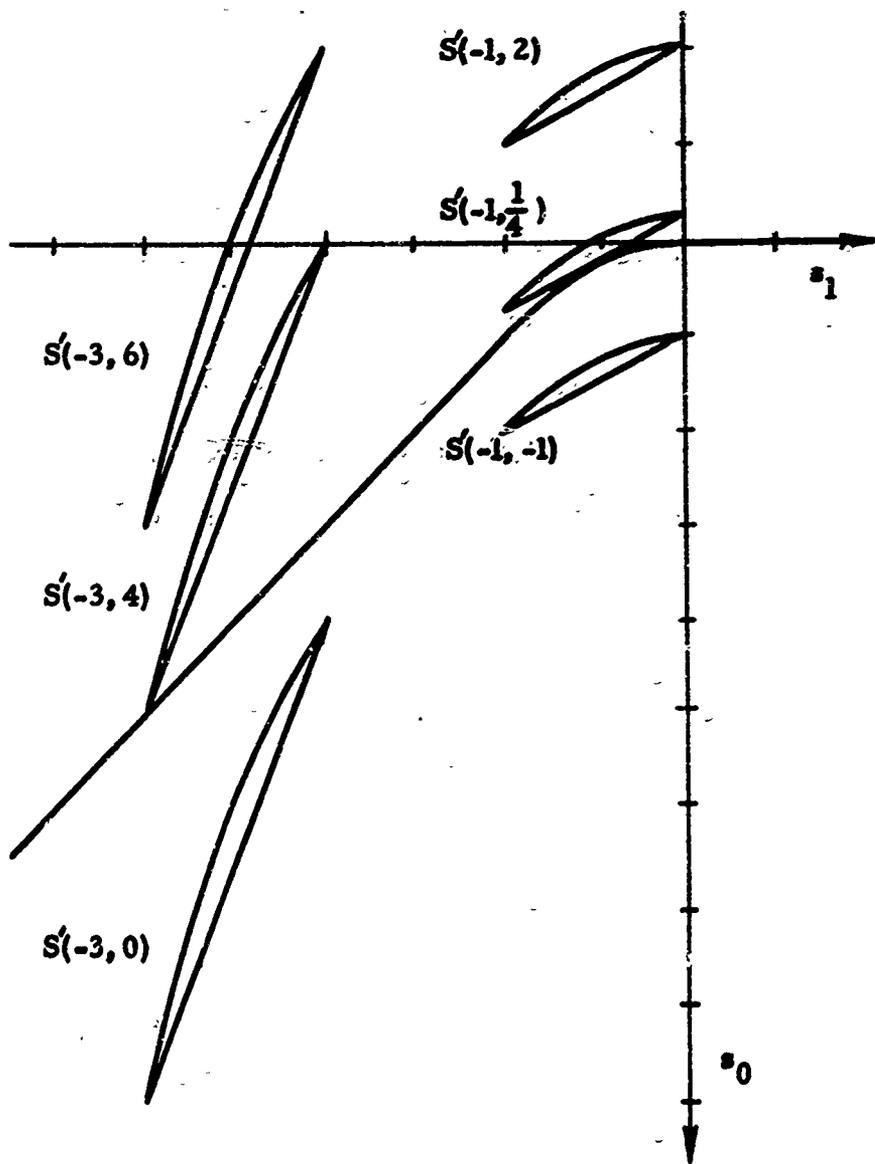


Figure 6-7. Mappings of R into S-space for  $z = -1$  and  $z = -3$

With the insight gained from the special cases, we may proceed to consider more general values of  $z$ . Note first that every tangent to the cross-section of the boundary of  $P_g^*$  at  $s_2 = 1$  corresponds to a point of  $C_g$ ; hence the minimizer uses only pure strategies. On the other hand, for each  $r_1$  corresponding to at least one  $\underline{r} \in R$  the image points  $\underline{s} \in S'(z, f)$  have the property that for fixed  $s_1$ , the value of  $s_0$  for  $r_2 = r_1$  is greater than or equal to  $s_0$  for  $r_2 = r_1^2$ . Therefore all optimal intersections of  $S(z, f)$  with  $P_g^*$  lie on the line corresponding to  $r_1 = r_2$  in  $R$ -space, and the maximizer always uses one of his extreme points  $u = 0$  or  $u = 1$ , or a mixture of these two points. For this reason we need only be concerned with the line segments in  $S'(z, f)$  given by

$$\begin{aligned} s_0 &= z^2 - f + (2z + 1)t \\ &\quad t \in [0, 1] \\ s_1 &= 2(z + t) \end{aligned} \tag{6.22}$$

in our analysis. Equations (6.22) may be written with  $t$  eliminated as

$$s_0 = (-f - z^2 - z) + (z + \frac{1}{2})s_1 \tag{6.23}$$

In the proofs in Chapter 5, the properties of simple algebraic maximization were emphasized. For variety, let us utilize here geometric properties of slope and support hyperplanes.

From Figure 6-5(b) it can be seen that the slope  $ds_0/ds_1$  of the boundary of  $P_g^*$  is between  $-1$  and  $0$ . Therefore if for given  $z$  the slope of the boundary line of  $S(z, f)$  is either less than  $-1$  or greater

than zero, we may be sure that the maximizer uses one of his pure end point strategies  $u = 0$  or  $u = 1$ . From (6.23),  $ds_0/ds_1|_{S(z,f)} = z + \frac{1}{2}$ . Hence,  $u$  uses pure strategies for  $z > -\frac{1}{2}$  or  $z < -\frac{3}{2}$ . For  $z > -\frac{1}{2}$ , (6.22) shows that  $s_0|_{\max}$  occurs for  $t = 1$  and that therefore  $s_1 > 0$  at the contact point of  $S(z, w)$  and  $P_S^*$ . It immediately follows that a separating plane is  $s_0 = 0$ . Substituting  $t = 1$  and  $s_0 = 0$  in (6.22) gives  $w(z) = f = z^2 + 2z + 1 = (z + 1)^2$ . Furthermore,  $t = 1$  gives  $F^0(u|z) = I_1(u)$ , and  $s_0 = 0$  for the separating plane gives  $G^0(v|z) = I_0(v)$ . These hold for  $z > -\frac{1}{2}$ .

If  $z < -\frac{3}{2}$ , then  $s_1 < -2$  from (6.22). In this region a support hyperplane and contact set with  $P_S^*$  is  $s_0 + s_1 = -1$ , implying  $G^0(v|z) = I_1(v)$ . The maximum for  $s_0$  is at  $t = 0$ . Since the contact point occurs on  $s_0 + s_1 + 1 = 0$ , we have  $z^2 - f + 2z + 1 = 0$ . Hence,  $w(z) = f = (z + 1)^2$  and  $F^0(u|z) = I_0(u)$ .

For the region  $z \in (-\frac{3}{2}, -\frac{1}{2})$ , the slope of (6.23) lies in  $(-1, 0)$ , and  $s_1 \in (-2, 0)$  for some values of  $t$  (See (6.22)). Therefore tangency of (6.23) with the curve  $s_0 - \frac{1}{4}s_1^2 = 0$  must be considered in determining the optimum payoff. The slopes of the two curves must be equal for tangency (and thus a separating plane) to occur. This requires

$$+s_1/2 = z + \frac{1}{2} \quad (6.24)$$

or

$$s_1 = +2z + 1 \quad (6.25)$$

at the point of contact. Using (6.22), this implies

$$t = \frac{1}{2}. \quad (6.26)$$

Hence  $F^0(u|z) = \frac{1}{2} I_0(u) + \frac{1}{2} I_1(u)$ , because we are working with the  $r_1 = r_2 = t$  boundary of R. On  $P_S^z$

$$s_0 = \frac{1}{4}(-s_1)^2 = z^2 + z + \frac{1}{4} \quad (6.27)$$

so that since

$$s_0 = z^2 - f + z + \frac{1}{2} \quad (6.28)$$

on  $S(z, f)$  in this region, eliminating  $s_0$  yields  $f = w(z) = \frac{1}{4}$ . The minimizer's pure strategy is concentrated at  $\tau = -\frac{(2z+1)}{2}$ ,

i. e.,  $G^0(v|z) = I_{-\frac{3}{2}-z-\frac{1}{4}}(v)$ .

The cases  $z = -\frac{3}{2}$  and  $z = -\frac{1}{2}$  are easily evaluated; the sets  $S'(-\frac{3}{2}, w(-\frac{3}{2}))$  and  $S'(-\frac{1}{2}, w(-\frac{1}{2}))$  are shown in Figure 6-8. For  $z = -\frac{1}{2}, w(-\frac{1}{2}) = \frac{1}{4}$ ,  $G^0(v|z) = I_0(v)$  and  $F^0(u|z) = \alpha I_0(u) + (1-\alpha) I_1(u)$  where  $\alpha \in [0, \frac{1}{2}]$ ; i. e., the maximizer has a choice of optimal strategies. Similarly, for  $z = -\frac{3}{2}, w(-\frac{3}{2}) = \frac{1}{4}$ ,  $G^0(v|z) = I_1(v)$ , and  $F^0(u|z) = \alpha I_0(u) + (1-\alpha) I_1(u)$  where  $\alpha \in [\frac{1}{2}, 1]$ .

The results in terms of  $z_N$  are summarized in Table 6-1.

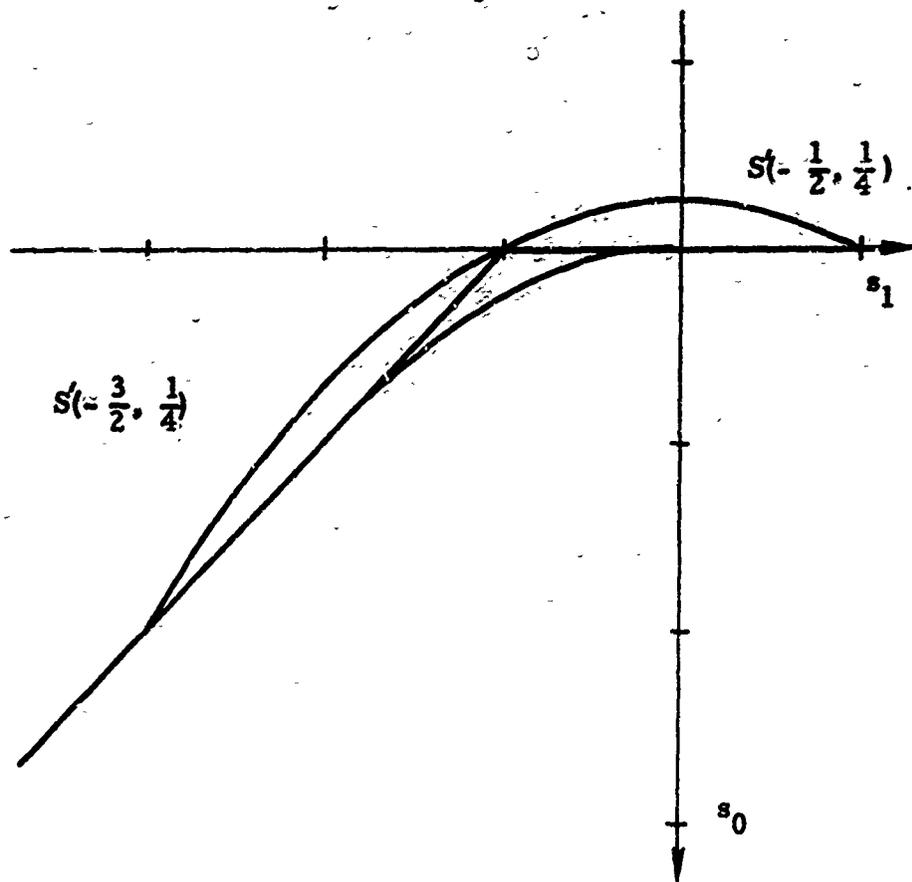


Figure 6-8. Sets tangent to  $P_S^*$ .

Table 6-1. Results for One Stage of Example 1

$z_N$	$F_{N-1}^0(u z_N)$	$G_{N-1}^0(v z_N)$	$w_N(z_N)$
$< -\frac{1}{2}$	$I_0(u)$	$I_1(v)$	$(z_N)^2$
$-\frac{1}{2}$	$\alpha I_0(u) + (1-\alpha)I_1(u)$ $\alpha \in [\frac{1}{2}, 1]$	$I_1(v)$	$\frac{1}{4}$
$-\frac{1}{2} \leq z_N < \frac{1}{2}$	$\frac{1}{2}I_0(u) + \frac{1}{2}I_1(u)$	$I_{-z_{N-1} + \frac{1}{2}}(v)$	$\frac{1}{4}$
$\frac{1}{2}$	$\alpha I_0(u) + (1-\alpha)I_1(u)$ $\alpha \in [0, \frac{1}{2}]$	$I_0(v)$	$\frac{1}{4}$
$> \frac{1}{2}$	$I_1(u)$	$I_0(v)$	$(z_N)^2$

The results may also be written in terms of  $u'_N$  and  $v'_N$  by the obvious transformations. Note that the payoff may be written

$$w_N(z_N) = \max [z_N^2, \frac{1}{4}]. \quad (6.29)$$

This is a piecewise quadratic as expected from the theory.

To find  $w_{N-1}(z_{N-1})$ , we repeat the basic processes above. Now, however, it is necessary to allow for the piecewise quadraticity of  $w_N(z_N)$ . Certainly for  $z_{N-1} \leq -\frac{3}{2}$  or  $z_{N-1} > \frac{3}{2}$  only the curve  $z_N^2$  is applicable, for the region  $z_N^2 < \frac{1}{4}$  is unattainable for any admissible controls  $u_{N-1} \in [0, 1]$ ,  $v_{N-1} \in [0, 1]$ . In this region, then, the results of Table 6-1 will apply with suitable changes in subscript.

When  $z_N \in [-\frac{1}{2}, \frac{1}{4}]$  is attainable, the situation is more complicated. There are several ways to argue concerning the establishment of the value and strategies for this region; one interesting technique is to use attainable-set arguments. Let us instead approximate the polynomial  $p(z) = \frac{1}{4}$  by the polynomial

$$p_\epsilon(z_N) = \epsilon z_N^2 + (1-\epsilon) \frac{1}{4} = \epsilon(z_N^2 - \frac{1}{4}) + \frac{1}{4} \quad \epsilon \in [0, 1] \quad (6.30)$$

Then as  $\epsilon \rightarrow 0$ ,  $p_\epsilon(z_N) \rightarrow p(z_N)$ . Also

$$\bar{w}_N(z_N) = \max [(-z_N)^2, \epsilon(z_N^2 - \frac{1}{4}) + \frac{1}{4}] \quad (6.31)$$

has the same points of discontinuity of  $d\bar{w}_N/dz_N$  as  $dw_N/dz_N$ . Let us evaluate the game  $p_\epsilon(z_N)$  given  $z_{N-1}$ . If  $w'_\epsilon(z_{N-1})$  denotes the value, then

$$w'_\epsilon(z_{N-1}) = \frac{1}{4} + \epsilon \min_{G(v)} \max_{F(u)} [1 \quad u \quad u^2] \begin{bmatrix} (z_{N-1}-1)^2 - \frac{1}{4} & 2(z_{N-1}-1) & 1 \\ 2(z_{N-1}-1) & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} \quad (6.32)$$

If we define  $z = z_{N-1} - 1$ , we see immediately that the portion of (6.32) of interest, i. e., the portion to be mini-maxed, is the same to within a bias constant as the one-stage problem (6.8). Therefore the strategies for the game  $p_\epsilon(z_{N-1})$  are independent of  $\epsilon$  and are the same as those for the game  $w_N(z_N)$ . The value is

$$w'_\epsilon(z_{N-1}) = \max \left[ \frac{1}{4} + \epsilon \left[ z_{N-1}^2 - \frac{1}{4} \right], \frac{1}{4} \right] \quad (6.33)$$

As  $\epsilon \rightarrow 0$ , it is clear that  $w_{\epsilon}(z_{N-1}) \rightarrow \frac{1}{4}$ . Suitable strategies for the limit game are, by continuity arguments similar to Lemma B, limits of the strategies for the game  $w_{\epsilon}(z_{N-1})$ , which we already noted are independent of  $\epsilon$ .

The game  $(z_{N-1})^2$  is precisely the same as  $w_N(z_N)$  except for subscripts, and has the same form of strategy. Thus  $(z_{N-1})^2$  and  $w_{\epsilon}(z_{N-1})$  have common optimal strategies, which may easily be read from Table 6-I. Either by inserting these strategies into (6.29) or by arguing concerning the continuity of the payoff and the fact that each branch of the game  $w_{N-1}(z_{N-1})$  is lower-bounded by  $\frac{1}{4}$ , we find that

$$w_{N-1}(z_{N-1}) = \max\left[z_{N-1}^2, \frac{1}{4}\right] \quad (6.34)$$

Noting that this is of the same form as (3.29) and that we have already argued that the optimal strategies are of the form in Table 6-I, we see that the multistage game is in fact solved, and in terms of the original definitions (6.1) the results may be summarized in Table 6-II.

Table 6-II. Results for Example 1

$z_i (i=1, \dots, N)$	$F_i^0(u_i^*   z_i)$	$G_i(v_i^*   z_i)$	$w_i(z_i)$
$z_i < -\frac{1}{2}$	$I_{-\frac{1}{2}}$	$I_{+\frac{1}{2}}$	$(z_i)^2$
$z_i = -\frac{1}{2}$	$\alpha I_{-\frac{1}{2}} + (1-\alpha)I_{\frac{1}{2}}^*$ $\alpha \in [\frac{1}{2}, 1]$	$I_{\frac{1}{2}}$	$\frac{1}{4}$
$-\frac{1}{2} < z_i < \frac{1}{2}$	$\frac{1}{2}I_{-\frac{1}{2}} + \frac{1}{2}I_{\frac{1}{2}}^*$	$I_{-z_i}$	$\frac{1}{4}$
$z_i = +\frac{1}{2}$	$\alpha I_{-\frac{1}{2}} + (1-\alpha)I_{\frac{1}{2}}^*$ $\alpha \in [0, \frac{1}{2}]$	$I_{-\frac{1}{2}}$	$\frac{1}{4}$
$z_i > \frac{1}{2}$	$I_{\frac{1}{2}}$	$I_{-\frac{1}{2}}$	$(z_i)^2$

## 6.2 COUNTER-EXAMPLE: A NON-POLYNOMIAL VALUE

As pointed out in Chapter 5, a polynomial game cannot be expected in general to have a value function which is a polynomial in  $z$ . A simple example will demonstrate this.

Suppose that  $u$ ,  $v$ , and  $z$  are scalars, that

$$J(z, u) = z^{2(N+1)} - u^2(N) \quad (6.35)$$

and that

\*These are optimal strategies. For  $i < N$  it may be shown that other optimal strategies also exist.

$$z(N+1) = z(N) + (z(N) + 1)u(N) + v(N) \quad (6.36)$$

We are interested in finding  $w_N(z(N))$ . Any other stages of the game are not of interest in this example. We assume that  $u(N) \in [0, 1]$ ,  $v(N) \in [0, 1]$ .

For ease of notation, certain subscripts may be dropped so that  $z = z(N)$ ,  $u = u(N)$ , and  $v = v(N)$ . The usual steps of substituting (6.36) into (6.35) and writing out the expression for  $w_N(z)$  give

$$w_N(z) = \max_{F(u|z)} \min_{G(v|z)} E[z^2 + 2z(z+1)u + 2zv + [(z+1)^2 - 1]u^2 + 2(z+1)uv + v^2] \quad (6.37)$$

In matrix notation, this is

$$0 = \max_{F(u|z)} \min_{G(v|z)} E \left\{ \begin{bmatrix} z^2 - w_N(z) & 2z & 1 \\ 2z(z+1) & 2(z+1) & 0 \\ (z+1)^2 - 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ u \\ u^2 \end{bmatrix} \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} \right\} \quad (6.38)$$

Using the moment definitions from the first example, (6.38) becomes

$$0 = \max_{r \in R} \min_{s \in S} [1 \ r_1 \ r_2] \begin{bmatrix} z^2 - w_N(z) & 2z & 1 \\ 2z(z+1) & 2(z+1) & 0 \\ (z+1)^2 - 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s_1 \\ s_2 \end{bmatrix} \quad (6.39)$$

Since the controls appear quadratically, the sets  $R$ ,  $S$ , and  $P_S^*$  are the same as those of Example 1. (Figures 6-1, 6-4, 6-5).

As in that example, form the sets

$$S'(z, f) = \{s_0, s_1 \mid \exists r \in R \ni s_0 = z^2 - f + 2z(z+1)r_1 + (z+1)^2 r_2 - r_2, s_1 = 2z + 2(z+1)r_1\} \quad (6.40)$$

and note that  $\underline{s} \in S(A(z), R, f)$  implies  $s_2 = 1$ , so that only a cross-section of  $P_S^*$  need be considered (Figure 6-5(b)).

Once again the minimizer will use pure strategies, whereas (because of the varying coefficient of  $r_2$  in the equation for  $s_0$ ) the maximizer may use either mixed or pure strategies. In  $S'(z, f)$ , the line  $r_1 = r_2$  generates a segment of

$$s_0 = z^2 - f + \frac{3z^2 + 4z}{2(z+1)} (s_1 - 2z) \quad (6.41)$$

Evaluating cases as before, we find that for  $\underline{s} \in S'(z, f)$ ,  $s_1 \geq 0$  for all  $r_1$  if  $z \geq 0$ . Therefore in this range  $G^0(v|z) = I_0(v)$  and (because the contact line is  $s_0 = 0$ )  $w(z) = 4z^2 + 4z$ . Furthermore, since  $r_1 = 1 = r_2$  is the best choice of moments for the maximizer,  $F^0(u|z) = I_1(u)$ . The strategy is arbitrary for  $z = 0$ .

If  $z \leq -1$ , then  $s_1 \leq -2$  and the intersection of  $S'(z, w(z))$  with  $P_S^*$  lies on the line  $s_0 + s_1 + 1 = 0$ . Therefore

$$f = z^2 + (2z^2 + 4z + 2)r_1 + z(z+2)r_2 + 2z + 1 \quad (6.42)$$

If  $z \leq -2$ , then clearly  $r_1 = r_2 = 1$  is optimum, yielding  $w(z) = 4z^2 + 8z + 3$ ,  $G^0(v|z) = I_1(v)$ , and  $F^0(u|z) = I_1(u)$ . If  $-2 < z < -1$ , then

the coefficients of  $r_1$  and  $r_2$  in (6.42) have opposite signs, suggesting a pure strategy solution for the maximizing player. Maximizing (6.42) over  $r_1 = t$ ,  $r_2 = t^2$  requires

$$t = -\frac{z^2 + 2z + 1}{z(z+2)}, \quad (6.43)$$

which after imposing the limits  $t \in [0, 1]$  implies

$$t = \begin{cases} 1 & z \leq -1 - \frac{\sqrt{2}}{2} \\ \frac{-(z+1)^2}{z(z+2)} & -1 > z \geq -1 - \frac{\sqrt{2}}{2} \end{cases} \quad (6.44)$$

Thus  $w(z) = 4z^2 + 8z + 3$ ,  $G^0(v|z) = I_1(v)$ , and  $F^0(u|z) = I_1(u)$  for  $z \leq -1 - \frac{\sqrt{2}}{2}$ . For  $-1 > z \geq -1 - \frac{\sqrt{2}}{2}$ , (6.42) and (6.44) imply that

$$w(z) = f = z^2 - \frac{(z+1)^4}{z(z+2)} + 2z + 1 = \frac{-(z+1)^2}{z(z+2)} \quad (6.45)$$

Also  $G^0(v|z) = I_1(v)$ , and  $F^0(u|z) = I_t(u)$ , where

$$t = \frac{-(z+1)^2}{z(z+2)}$$

For  $-1 < z < 0$ , examination of (6.40) reveals that the coefficient of  $r_2$  is negative, implying that the maximizer will use pure strategies. Parameterizing  $S'(z, w(z))$  by  $r_1 = t$ ,  $r_2 = t^2$  and inserting in the equation (See Figure 6-5(b)) for the boundary of  $P_S^*$

$$s_0 = z^2 - f + 2z(z+1)t + z(z+2)t^2 = (z^2 + 2z(z+1)t + (z+1)^2 t^2) = \left(\frac{-z-1}{2}\right)^2 \quad (6.46)$$

Hence

$$f = -t^2 \quad (6.47)$$

Here  $t = 0$  is the obvious choice; i.e.,  $F^0(u|z) = I_0(u)$ , in this region. The intersection point with  $P_S^*$  has  $s_1 = 2z$ , implying the pure strategy  $G^0(v|z) = I_{-z}(v)$  for the minimizer. From (6.47) it is clear that  $w(z) = 0$ . Table 6-III summarizes the solution and Figure 6-9 shows representative  $S'(z, w(z))$  sets. Of particular interest is that for  $z \in [-1 - \frac{\sqrt{2}}{2}, -1]$   $w(z)$  is rational but not a polynomial. Therefore, if a further stage is to be solved, the method of dual cones is unlikely to be applicable.

Table 6-III. Solutions for Example 2

$z$	$F^0(u z)$	$G^0(v z)$	$w(z)$
$z \leq -1 - \frac{\sqrt{2}}{2}$	$I_1(u)$	$I_1(v)$	$(2z+3)(2z+1)$
$z > -1 - \frac{\sqrt{2}}{2}, z \leq -1$	$I_t(u)$ $t = \frac{-(z+1)^2}{z(z+2)}$	$I_1(v)$	$\frac{-(z+1)^2}{z(z+2)}$
$z > -1, z < 0$	$I_0(u)$	$I_{-z}(v)$	0
0	Arbitrary	$I_0(v)$	0
$z > 0$	$I_1(u)$	$I_0(v)$	$4z(z+1)$

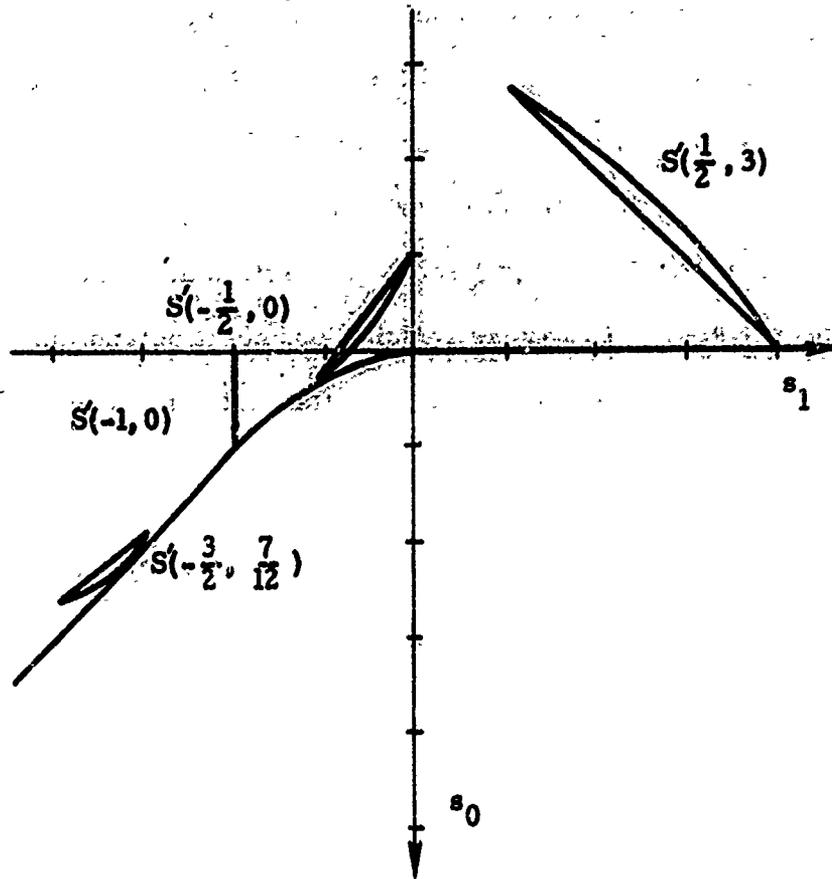


Figure 6-9. Representative sets  $S(z, w)$  for Example 2.

### 6.3 A SIMPLE PROBLEM WITH VECTORS

The biggest obstacle to finding solutions of a non-numerical nature is dimensionality, for spaces larger than three-dimensional are almost impossible to visualize. The following problem is of small enough dimension to be pictured and still is an interesting problem containing vectors.

Let  $\underline{z}$  and  $\underline{u}$  be two-dimensional and let  $v$  be a scalar for a system with dynamics

$$\begin{aligned} z_1(i+1) &= z_1(i) + u_1(i) - \frac{\sqrt{2}}{2} u_2(i) + \frac{\sqrt{2}}{2} v(i), \\ z_2(i+1) &= z_2(i) + \frac{\sqrt{2}}{2} u_2(i) + \frac{\sqrt{2}}{2} v(i), \end{aligned} \quad (6.48)$$

and with  $v(i) \in [0, 1]$ ,  $u_1(i) \in [0, 1]$ ,  $u_2(i) \in [0, 1]$ . For the payoff function choose

$$J = z_1^2(N+1) + z_2^2(N+1) - u_1^2(N) - u_2^2(N) \quad (6.49)$$

As in the previous examples, drop the stage indices after substituting (6.48) into (6.49) and use vector-matrix form for  $J$  to get

$$w(z_1, z_2) = \min_{G(v)} \max_{F(u)} E [1 \ u_1 \ u_2 \ u_1 u_2] \begin{matrix} \left[ \begin{array}{cc} z_1^2 + z_2^2 & \sqrt{2}(z_1 + z_2) \\ 2z_1 & \sqrt{2} \\ \sqrt{2}(z_2 - z_1) & 0 \\ -\sqrt{2} & 0 \end{array} \right] \begin{bmatrix} 1 \\ 1 \\ v \\ -2 \\ 0 \end{bmatrix} \end{matrix} \quad (6.50)$$

Using the usual definitions, this may be rewritten

(6.51)

$$0 = \min_{\underline{s} \in S} \max_{\underline{r} \in R} [1 \ r_1 \ r_2 \ r_x] \begin{bmatrix} z_1^2 + z_2^2 - w(z) & \sqrt{2}(z_1 + z_2) & 1 \\ 2z_1 & \sqrt{2} & 0 \\ \sqrt{2}(z_2 - z_1) & 0 & 0 \\ -\sqrt{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s_1 \\ s_2 \end{bmatrix}$$

The set  $S$  is the same as in example 1, as is  $P_S^*$ . We see that the mapping  $S(A(z), R, f)$  once again has  $s_2 = 1$ , so that Figure 6-5(b) is again usable.

The set  $R$  may be constructed by forming the set

$\hat{C}_R = \{\underline{r} \mid r_1 = t_1, r_2 = t_2, r_x = t_1 t_2, t_i \in [0, 1]\}$  and then taking its closure. The sets  $\hat{C}_R$  and  $\hat{R}$  are shown in Figure 6-10, where  $\hat{C}_R$  and  $\hat{R}$  are projections for  $r_0 = 1$  of  $C_R$  and  $R$ .

The interesting thing about  $\hat{R}$  is that it is a tetrahedron and has as its vertices the points  $(r_1, r_2, r_x) = (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1)$ . These points correspond to pure strategies  $I_{u_1, u_2}(\underline{u}) = I_{00}(\underline{u}), I_{10}(\underline{u}), I_{01}(\underline{u}), I_{11}(\underline{u})$  respectively.

The set  $S'(\underline{z}, f)$ , which is the projection on  $s_2 = 1$  of the image of  $R$  for a given parameter  $f$  and initial state  $\underline{z}$  is defined by

(6.52)

$$S'(\underline{z}, f) = \{s_0, s_1 \mid s_0 = z_1^2 + z_2^2 - f + 2z_1 r_1 + \sqrt{2}(z_2 - z_1) r_2 - \sqrt{2} r_x, \\ s_1 = \sqrt{2}(z_1 + z_2) + \sqrt{2} r_1, \underline{r} \in R\}$$

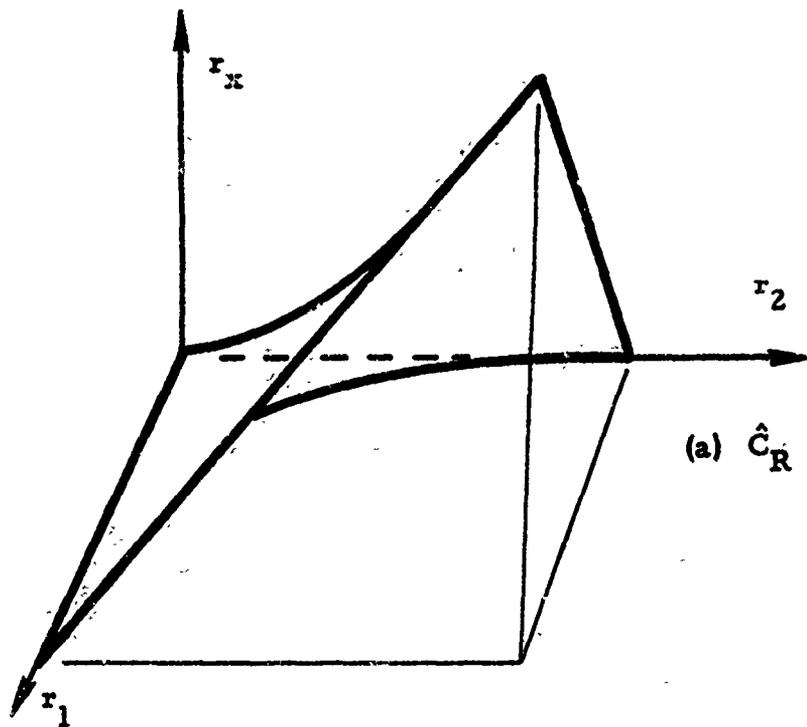
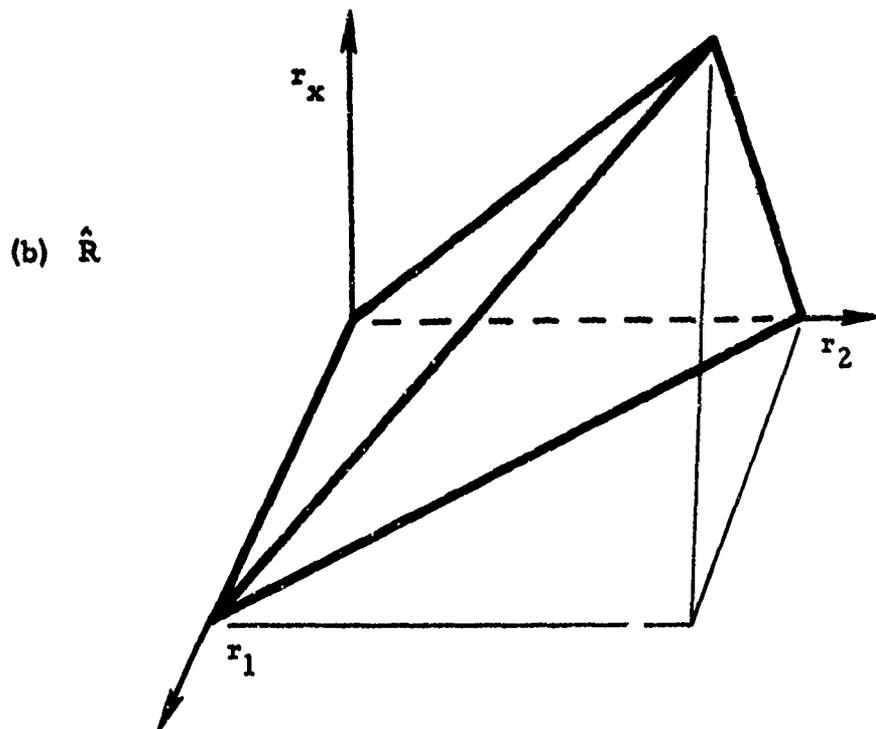


Figure 6-10. The sets  $\hat{C}_R$  and  $\hat{R}$  for Example 3.



We will consider the interactions of this set with  $P_S^*$  for various values of  $\underline{z}$ . Note that the maximizer's moments  $r_1$  and  $r_2$  may be chosen independently, provided that the coupling  $r_x$  is accounted for.

Case 1:  $z_1 + z_2 \geq 0$ . In this region,  $s_1 = \sqrt{2}(z_1 + z_2) + \sqrt{2} r_1 \geq 0$  for all admissible  $r_1$ , implying the pure strategy  $G^0(v|\underline{z}) = I_0(v)$  for the minimizer and

$$f = z_1^2 + z_2^2 + 2z_1 r_1 + \sqrt{2}(z_2 - z_1) r_2 - \sqrt{2} r_x \quad (6.53)$$

as the expression to be maximized over  $\underline{r} \in R$ .

For  $z_1 < 0$ ,  $r_1 = 0$  is obvious, as is  $r_2 = 0$  for  $z_2 - z_1 < 0$ . In both cases  $r_x = 0$  follows from the choice of  $r_1$  or  $r_2$ . If both  $z_1 > \frac{\sqrt{2}}{2}$  and  $(z_2 - z_1) > 1$ , then clearly the penalty of taking  $r_x = 1$  is worth the benefit from having both  $r_1 = 1$  and  $r_2 = 1$ . If, however, we have  $z_1 > 0$ ,  $z_2 > z_1$ , but either  $z_1 < \frac{\sqrt{2}}{2}$  or  $z_2 < 1 + z_1$ , then further examination is necessary to determine the desired strategy.

Figure 6-11 shows the form of  $S'(\underline{z}, f)$  for  $\underline{z}$  in this region. The corner markings indicate the points of  $R$  which generate the corners.

From this it is clear that  $I_{01}$  or  $I_{10}$  will be preferred depending upon which has the larger coefficient. Thus

$$2z_1 > \sqrt{2}(z_2 - z_1) \quad (6.54)$$

or

$$(\sqrt{2}+1)z_1 > z_2$$

leads to choice of  $I_{10}$ , the opposite inequality leads to the opposite choice, and equality implies an arbitrary mixture of the two strategies. Results for Case 1 are summarized in Figure 6-12.

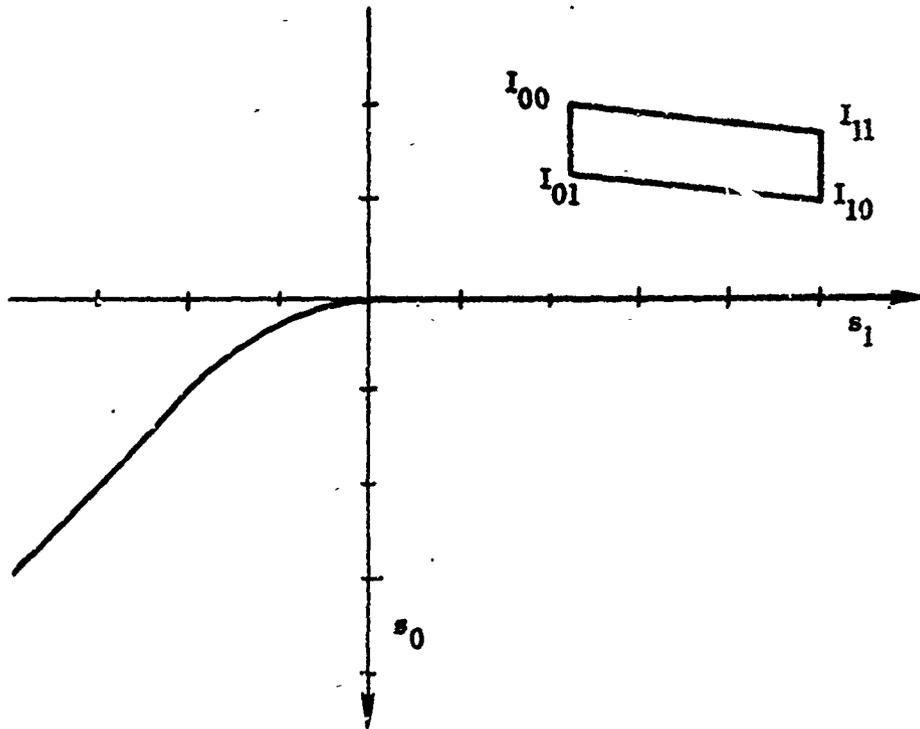


Figure 6-11. Representative mapping of  $S'(\underline{z}, f)$

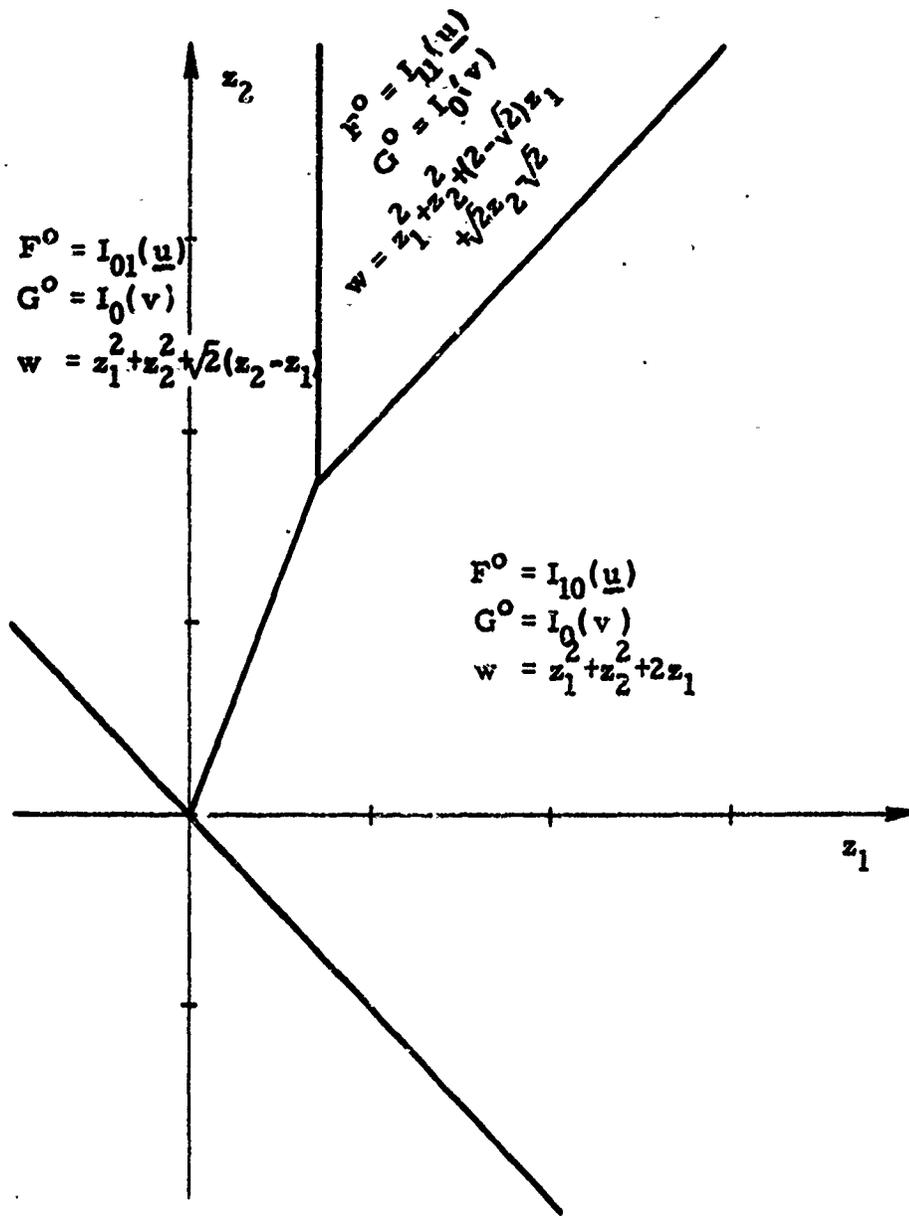


Figure 6-12. Strategies and values for Case 1.

Case 2: If  $s_1 = \sqrt{2}(z_1 + z_2) + \sqrt{2} \leq -2$ , the minimizer uses the pure strategy  $I_1(v|\underline{z})$  and the intersection of  $S'(\underline{z}, w(\underline{z}))$  with  $P_S^*$  lies on the line  $s_0 + s_1 + 1 = 0$ . From this it follows that

$$f = z_1^2 + z_2^2 + (2z_1 + \sqrt{2})r_1 + \sqrt{2}(z_2 - z_1)r_2 - \sqrt{2}r_x + \sqrt{2}(z_1 + z_2) + 1 \quad (6.55)$$

Arguing as in Case 1, we find the results which are summarized in Figure 6-13.

Case 3:  $z_1 + z_2 < 0$ ,  $z_1 + z_2 > -1 - \sqrt{2}$ . This final region is more involved to evaluate because the curved nature of the boundary of  $P_S^*$  ( $s_0 = (-\frac{s_1}{2})^2$ ) in part of this area makes possible non-trivial mixed strategies for the maximizer and fractional pure strategies for the minimizer.

Note that on  $S'(\underline{z}, f)$  for  $r_1 \neq 0$  we can relate  $s_0$  and  $s_1$  by

$$\begin{aligned} s_0 &= z_1^2 + z_2^2 - f + 2z_1 \left( \frac{s_1}{\sqrt{2}} - z_1 - z_2 \right) + \sqrt{2}(z_2 - z_1)r_2 - \sqrt{2}r_x \\ &= -z_1^2 - 2z_1z_2 + z_2^2 - f + \sqrt{2}z_1s_1 + \sqrt{2}(z_2 - z_1)r_2 - \sqrt{2}r_x \end{aligned} \quad (6.56)$$

For  $r_1 = 0$  or  $r_1 = 1$ ,  $s_1$  is constant.

Consider  $z_1 \geq 0$ , so that  $z_2 - z_1 \leq 0$ . Then the mapping (6.52) of  $R$  is of the form shown in Figure 6-14.

Clearly  $I_{10}(\underline{u})$  is preferred by the maximizer, and contact with  $P_S^*$  occurs on the curve  $s_0 = \frac{1}{4} s_1^2$  for  $(z_1 + z_2) \leq -1$  and on  $s_0 = 0$  for  $z_1 + z_2 > -1$ . The strategy for the minimizer is  $I_t(v)$ , where  $t = -\frac{s_1}{2} = -\frac{\sqrt{2}}{2}(z_1 + z_2 + 1)$  in the former region and  $I_0$  in the latter, with the payoff function evaluated accordingly as either

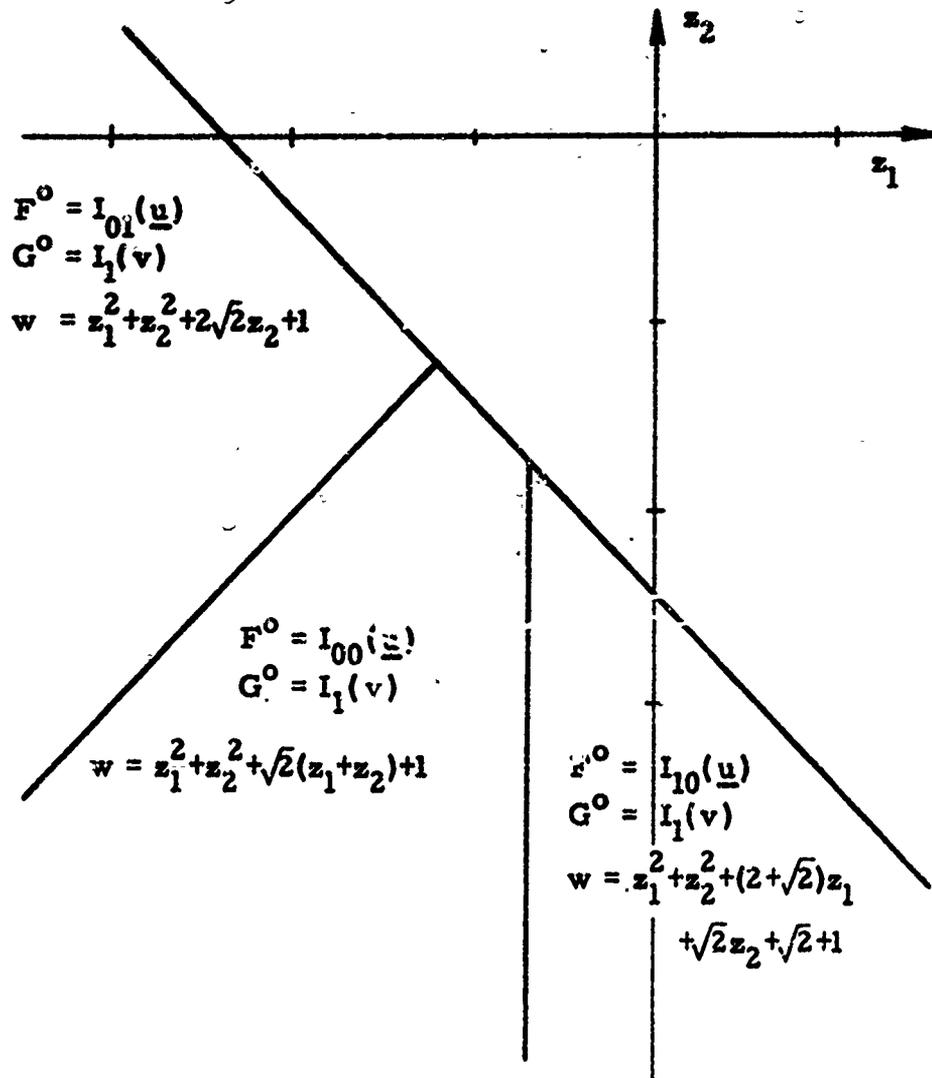


Figure 6-13. Strategies and values for Case 2.

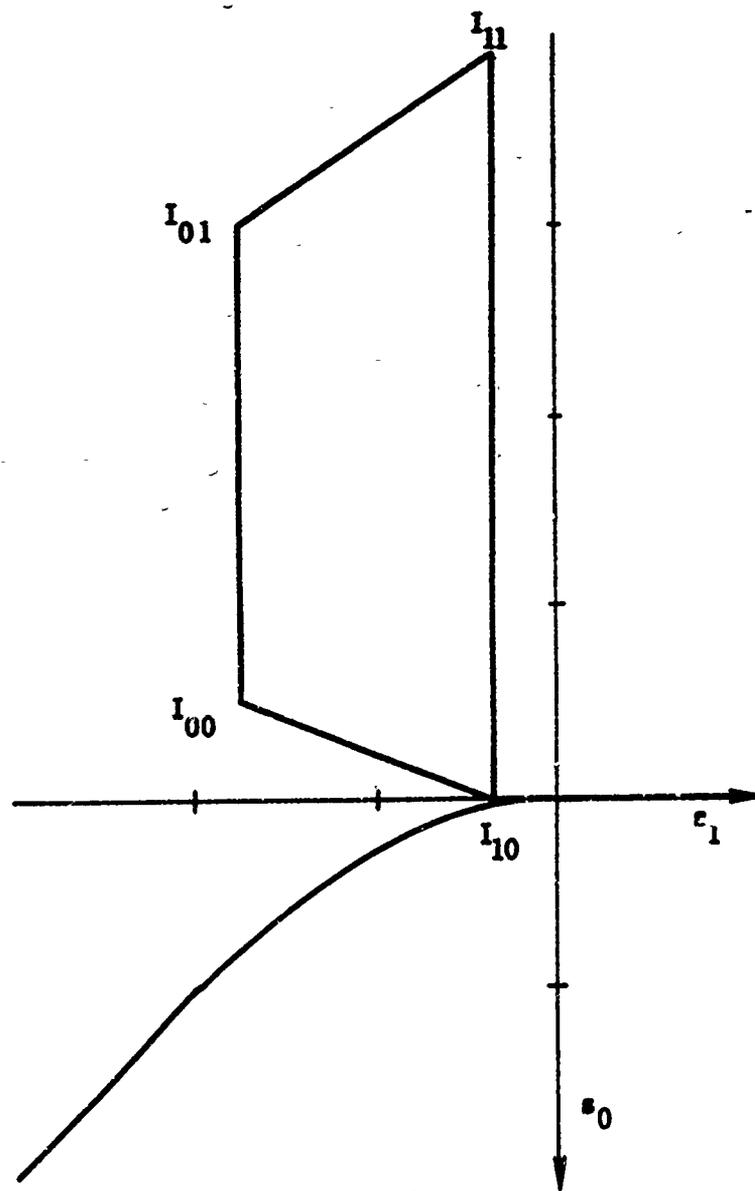


Figure 6-14. Mapping of R for  $z_1 \geq 0$

$$w(\underline{z}) = \begin{cases} +\frac{z_1^2}{2} + \frac{z_2^2}{2} + z_1 - z_1 z_2 - z_2 - \frac{1}{2} & z_1 + z_2 < -1 \\ z_1^2 + z_2^2 + 2z_1 & z_1 + z_2 > -1 \end{cases} \quad (6.57)$$

If  $z_1 < 0$ , more possibilities arise. Let us consider the case  $z_2 - z_1 < 0$  in some detail. A possible configuration of the mapping of R is shown in Figure 6-15.

From (6.56) we know that the slope of the line from  $I_{00}$  to  $I_{10}$  is  $\sqrt{2} z_1$ . Since the slope of the  $P_S^*$  boundary is greater than -1 and less than 0,  $z_1 < -\frac{\sqrt{2}}{2}$  implies that  $I_{00}$  is the contact point, with suitable interpretations as in the case  $z_1 > 0$ .

On the other hand  $z_1 \geq -\frac{\sqrt{2}}{2}$  implies a contact point either at  $I_{10}$  or on the line from  $I_{00}$  to  $I_{10}$ , depending upon the exact values involved. For the line to be tangent to the curve  $s_0 = \frac{1}{4} s_1^2$ , the slopes must be the same at the point of contact. This implies that

$$\frac{s_1}{2} = \sqrt{2} z_1$$

This equation along with the definition of  $s_1$  on the set  $S(\underline{z}, f)$  gives

$$s_1 = 2\sqrt{2} z_1 = \sqrt{2}(z_1 + z_2) + \sqrt{2} r_1$$

or

$$r_1 = (z_1 - z_2) \quad (6.58)$$

Since  $0 \leq r_1 \leq 1$ , the limits of the range of internal contact are clear.

Where it applies, the mixed strategy for player I is

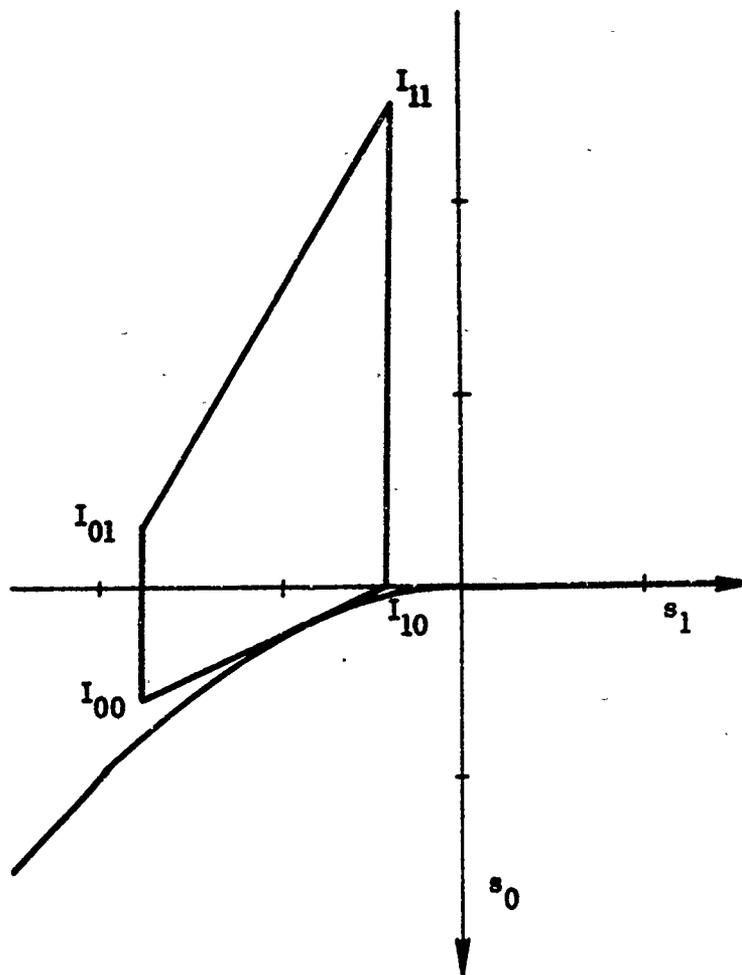


Figure 6-15. Mapping of R for  $z_2 < z_1 < 0$

$$F^0(\underline{u}|\underline{z}) = (1 + z_2 - z_1) I_{00}(\underline{u}) + (z_1 - z_2) I_{10}(\underline{u}) \quad (6.59)$$

and the minimizer strategy is

$$G^0(v|\underline{z}) = I - \sqrt{2} z_1 (v) \quad (6.60)$$

The value is

$$w(\underline{z}) = (z_1 - z_2)^2 \quad (6.61)$$

If  $r_1$  is limited, the results are obvious.

Similarly, if  $z_2 > z_1$  and  $z_1 < 0$ , the mapping has the appearance of Figure 6-16.

In this case, the line of interest is from  $I_{01}$  to  $I_{10}$  and has equation

$$s_0 = z_1^2 + z_2^2 - f + (2z_1 + \sqrt{2} z_1 - \sqrt{2} z_2) \left( \frac{s_1}{\sqrt{2}} - z_1 - z_2 \right) + \sqrt{2} (z_2 - z_1) \quad (6.62)$$

In the region of interest, the slope of this line is less than -1 and therefore  $I_{01}$  is the preferred strategy. A region for which tangency is possible requires  $z_1 > z_2$ , which violates the hypothesis for the region. The results for Case 3 are summarized in Figure 6-17.

A comment on the nature of the continuity of the results is perhaps in order. Within regions, of course, continuity is obvious. At boundaries of regions, however, the continuity is not always so clear. This is because only upper semi-continuity holds; that is, if  $D$  is a sufficiently small open set containing the set of optimal

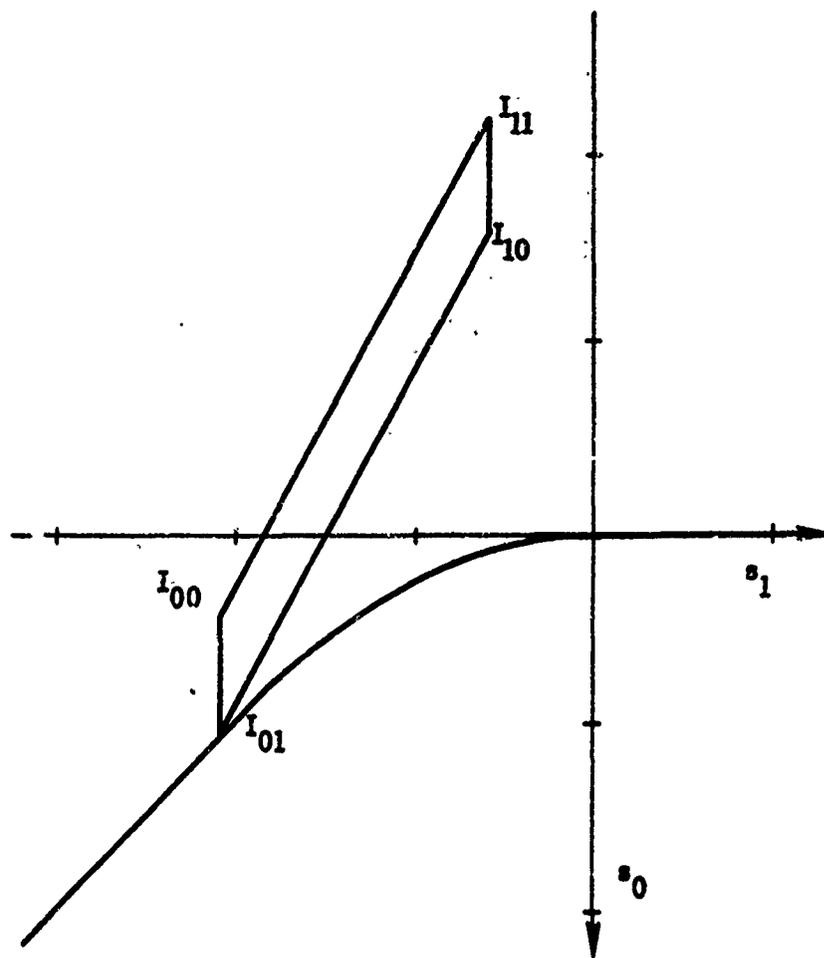


Figure 6-16. Mapping of R for  $s_1 < 0$ ,  $z_2 > z_1$

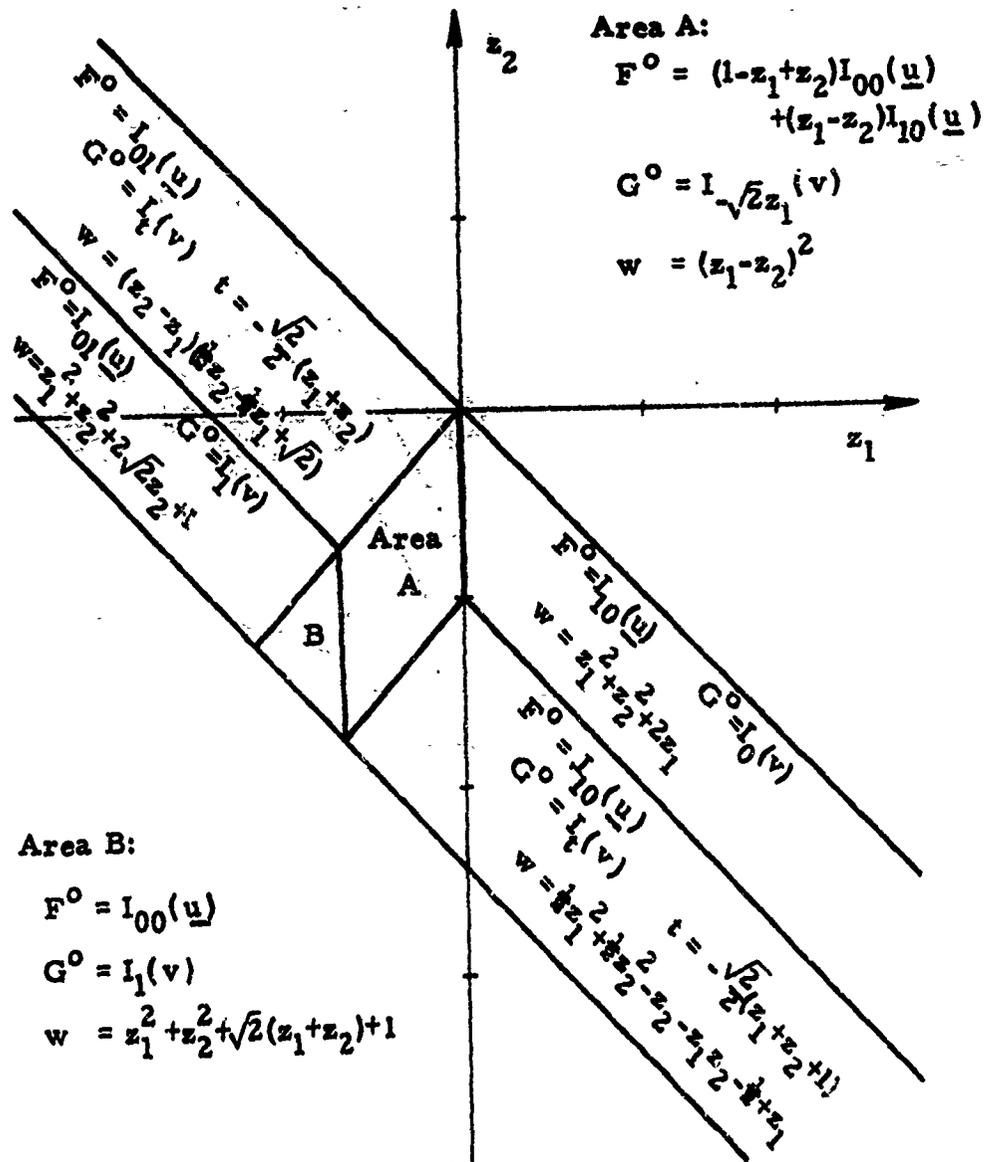


Figure 6-17. Strategies and values for Case 3.

strategies  $R^0$  at a point  $\underline{z}$ , then for  $\underline{z}'$  sufficiently close to  $\underline{z}$ , the optimal strategies at  $\underline{z}'$  are contained in  $D$ . However,  $R^0$  may not be contained in the set of optimal strategies of  $\underline{z}'$ . The meaning of this for the boundary regions is that strategies there are typically not unique. Thus solutions on opposite sides of the boundary may not be near each other although both are near some optimal strategy for the boundary point.

For example, consider the Region A boundary  $z_1 = -\frac{\sqrt{2}}{2}$  in Figure 6-17. The situation here is as sketched in Figure 6-18. From this it can be seen that any strategy

$$F(\underline{u}) = (1 - \alpha) I_{00}(\underline{u}) + \alpha I_{01}(\underline{u}), \quad \alpha \in [0, z_2 + \frac{\sqrt{2}}{2}] \quad (6.63)$$

will be optimal for the maximizer. Strategies on both sides of the line  $z_1 = -\frac{\sqrt{2}}{2}$  are continuous with this strategy for some  $\alpha$ .

Figures 6-19 and 6-20 are sketches of the results given in detail in Figures 6-12, 6-13, and 6-17.

#### 6.4 LINEAR PROGRAMMING FOR APPROXIMATE SOLUTIONS

Chapter 4 discussed the use of linear programming to generate approximate solutions to game problems. We shall see some of the implications of the technique in an example. Only a simple problem evaluated at a single data point is needed to clarify the ideas.

Consider the game of Example 1, Section 6.1, with one stage to go and with initial condition  $z_N = 0$ . From Equation (6.8) we have

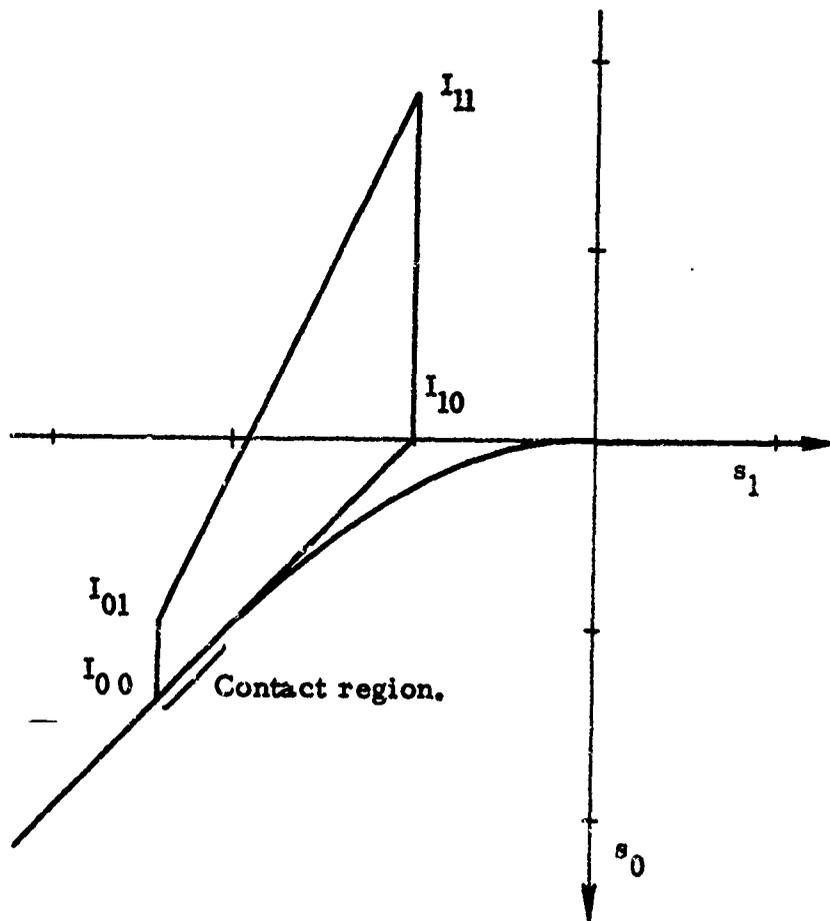


Figure 6-18. Case of non-unique strategies.

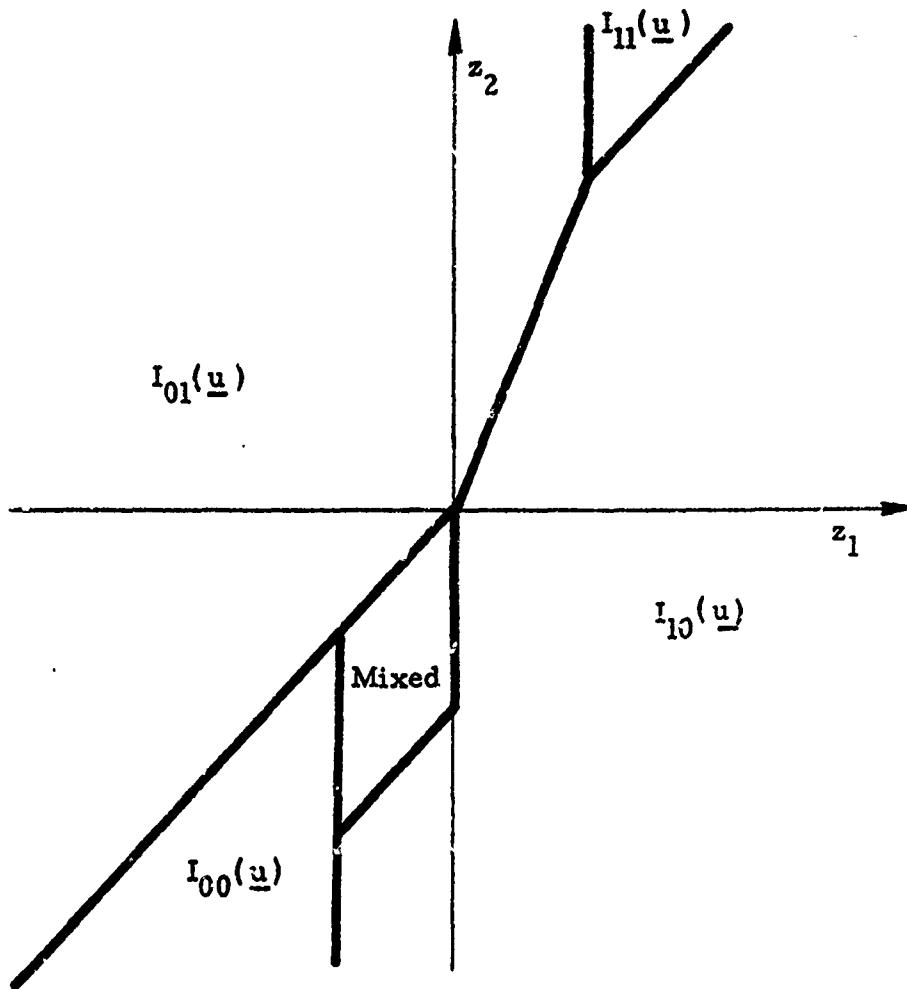


Figure 6-19. Optimal strategy regions for maximizing player.

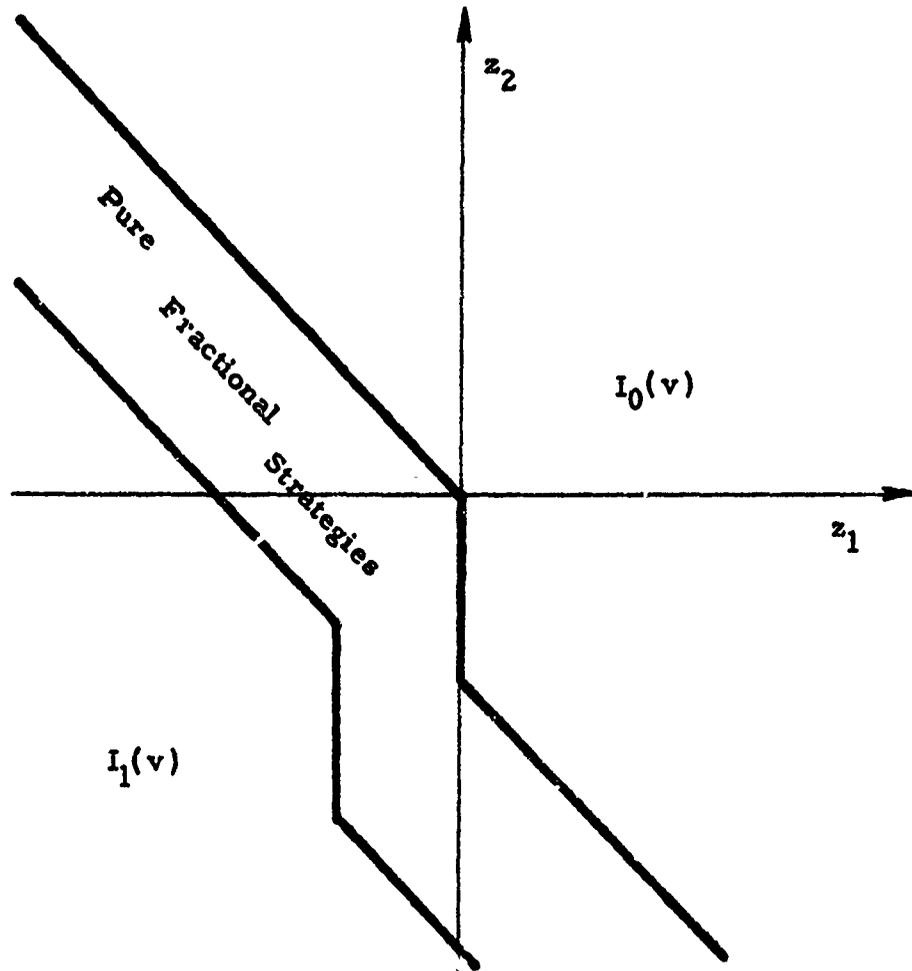


Figure 6-20. Optimal strategy regions for minimizing player.

$$w(-1) = \min_{G(v)} \max_{F(u)} E \left\{ [1 \ u \ u^2] \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} \right\} \quad (6.64)$$

In Section 6.1 the solution was found to be

$$w(-1) = \frac{1}{4}$$

$$F^0(u) = \frac{1}{2} I_0 + \frac{1}{2} I_1 \quad (6.65)$$

$$G^0(v) = I_{\frac{1}{2}}$$

The set  $\bar{R}$  is shown in Figure 6-1. Let us approximate it by the polygon  $\bar{R}$  shown in Figure 6-21.

To lie within this polygon,  $\underline{r}$  must satisfy

$$r_2 \leq r_1$$

$$r_2 \geq \frac{1}{4} r_1$$

$$r_2 \geq \frac{3}{4} r_1 - \frac{1}{8} \quad (6.66)$$

$$r_2 \geq \frac{5}{4} r_1 - \frac{3}{8}$$

$$r_2 \geq \frac{7}{4} r_1 - \frac{3}{4}$$

The polygon is internal to  $R$  and thus our solution point  $\underline{r}$  of the approximate problem will be a viable strategy for the maximizer.

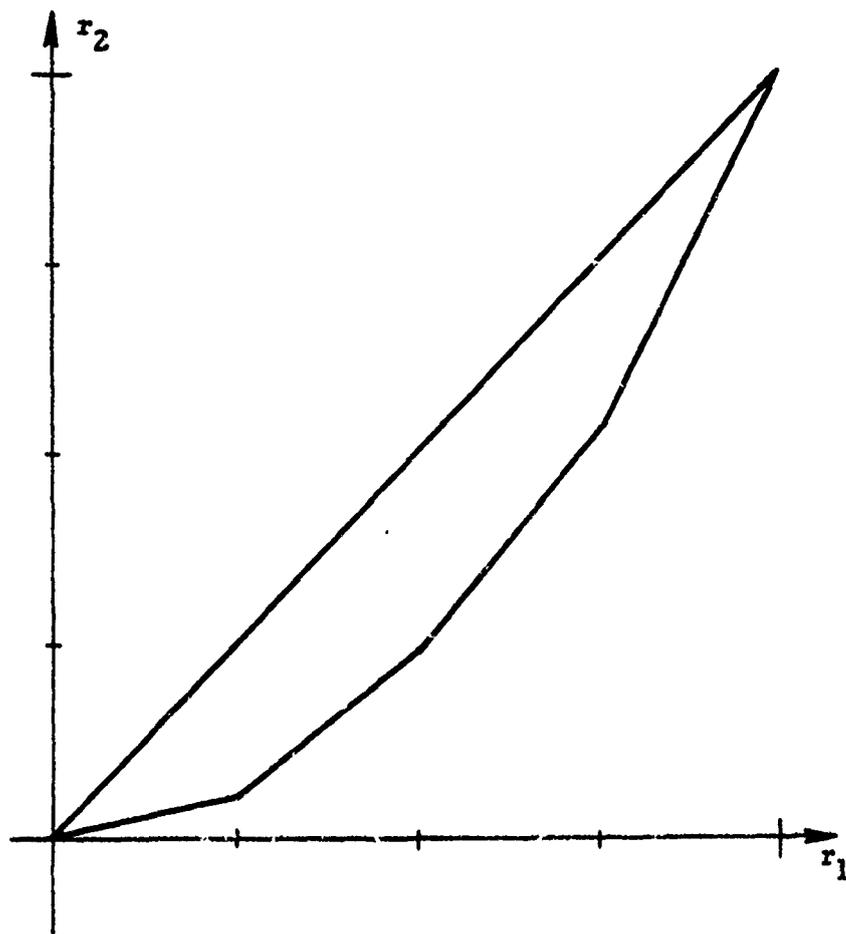


Figure 6-21. Polygonal approximation to  $\hat{R}$ .

Now create an approximation  $\hat{P}_3^*$  to  $P_S^*$  by using the support planes generated by points in  $\hat{C}_S$  (Theorem 4.6). A plane will have the general form  $\{\underline{s} | s_0 + ts_1 + t^2 s_2 = 0, t \in [0, 1]\}$ . Let us choose  $t = 0, \frac{1}{6}, \frac{1}{4}, \frac{2}{5}, \frac{5}{8}, \frac{6}{7}, 1$ . Also note that we are interested only in  $s_2 = 1$  because of the transformation matrix in (6.64). Thus we say that if  $\underline{s} \in \hat{P}_3^*$ , then  $s_0, s_1$  must satisfy

$$\begin{aligned}
 s_0 &\geq 0 \\
 s_0 + \frac{1}{6} s_1 &\geq -\frac{1}{36} \\
 s_0 + \frac{1}{4} s_1 &\geq -\frac{1}{16} \\
 s_0 + \frac{2}{5} s_1 &\geq -\frac{4}{25} \\
 s_0 + \frac{5}{8} s_1 &\geq -\frac{25}{64} \\
 s_0 + \frac{6}{7} s_1 &\geq -\frac{36}{49} \\
 s_0 + s_1 &\geq -1
 \end{aligned}
 \tag{6.67}$$

However, after using the usual biasing parameter  $f$ , we find from (6.64) that  $s_0, s_1$  must also satisfy

$$\begin{aligned}
 s_0 &= 1 - f - 2r_1 + r_2 \\
 s_1 &= -2 + 2r_1
 \end{aligned}
 \tag{6.68}$$

Substituting this in (6.67), rewriting (6.66), and maximizing  $f$ , we find that we have the following linear programming problem:

maximize  $f$   
 $(r_1, r_2, f)$   
 subject to

$$\begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{4} & 1 & 0 \\ -\frac{3}{4} & 1 & 0 \\ -\frac{5}{4} & 1 & 0 \\ -\frac{7}{4} & 1 & 0 \\ -2 & 1 & -1 \\ -\frac{5}{3} & 1 & -1 \\ -\frac{3}{2} & 1 & -1 \\ -\frac{6}{5} & 1 & -1 \\ -\frac{3}{4} & 1 & -1 \\ -\frac{2}{7} & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ f \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{8} \\ -\frac{3}{8} \\ -\frac{3}{4} \\ -1 \\ -\frac{25}{36} \\ -\frac{9}{16} \\ -\frac{9}{25} \\ -\frac{9}{64} \\ -\frac{1}{49} \\ 0 \end{bmatrix} \quad (6.69)$$

For this problem, the solution is  $r_1^0 = r_2^0 = \frac{39}{80}$  and  $f^0 = \frac{21}{80}$ . Thus  $w = \frac{21}{80}$  and  $F^0(u) = \frac{41}{80} I_0(u) + \frac{39}{80} I_1(u)$ . Equality of the constraints holds in the first, ninth, and tenth of (6.69). The latter two correspond to the hyperplanes generated by  $t = \frac{2}{5}$  and  $t = \frac{5}{8}$ . It should be noted that neither of the latter planes is a separating hyperplane of  $F_S^*$  and the mapping of  $\bar{R}$  (See Figure 6-22),

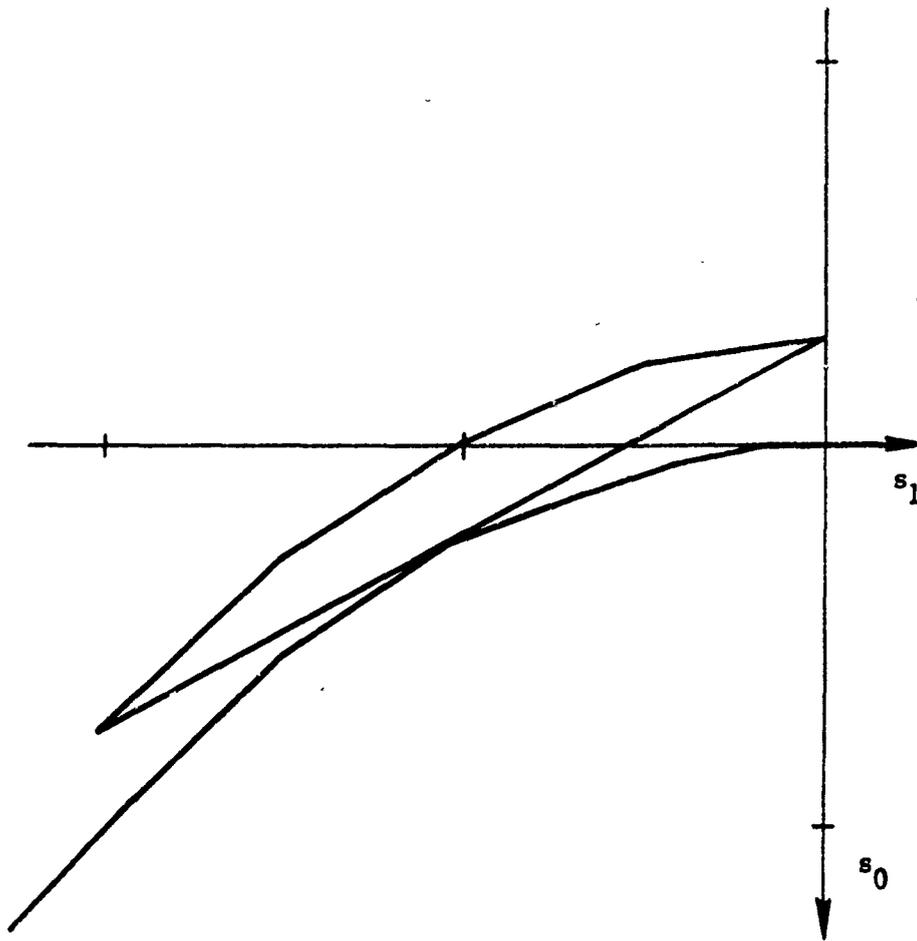


Figure 6-22. Polyhedra at optimum payoff point.

although each supports  $P_3^*$ . Either (or a combination of both) may be used as an approximate strategy for the minimizer, since it is known that pure strategies are sufficient for him.

If another iteration is used, with the R approximation being the same but with  $P_3^*$  approximated using  $t = 0, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, 1$  (so that a smaller granularity appears in the region of the possible solution  $t = \frac{2}{5}, t = \frac{5}{8}$  from the first iteration), it is found that  $w = f^0 = \frac{1}{4}$  and that both  $r_1^0 = r_2^0 = \frac{9}{20}$  and  $r_1^0 = r_2^0 = \frac{11}{20}$  yield this value (as will  $r_1^0 = r_2^0, r_1^0 \in [\frac{9}{20}, \frac{11}{20}]$ ). Support planes  $t = \frac{2}{5}, t = \frac{1}{2}$  give the latter  $\underline{r}$  values and  $t = \frac{1}{2}, t = \frac{3}{5}$  give the former. In this case  $t = \frac{1}{2}$  is a separating hyperplane and  $I_1(v)$  is a good strategy for the minimizer. Either or both of the  $\underline{r}$ -moments may be used by the maximizer with justification; one suitable c. d. f. is  $F^0(u) = \frac{9}{20} I_0(u) + \frac{11}{20} I_1(u)$ . Closer approximations achieved by smaller granularity are of course possible.

## CHAPTER 7

### COMMENTS ON DUAL CONES FOR DIFFERENTIAL GAMES

Two-person zero-sum differential games with closed-loop strategies have been the subject of considerable research interest, and we would be remiss if we did not consider extending our results to such games. We shall find that this extension seems fraught with peril, however, and therefore confine ourselves to comments and to formal arguments. Open-loop strategies are somewhat simpler, but many of the same comments apply.

#### 7.1 THE PROBLEM OF DIFFERENTIAL GAMES

The differential game analog of our multistage games has dynamics

$$\dot{\underline{z}}(t) = \underline{f}(\underline{z}(t), \underline{u}(t), \underline{v}(t), t) \quad (7.1)$$

and payoff function

$$J(\underline{z}(\tau); \underline{u}(t), \underline{v}(t); T, \tau) = g_f(\underline{z}(T)) + \int_{\tau}^T g(\underline{z}(t), \underline{u}(t), \underline{v}(t), t) dt \quad (7.2)$$

where  $\underline{z}(\tau)$  is an initial condition given at time  $\tau$  for the dynamics equation (7.1), and  $\underline{u}$  and  $\underline{v}$  are control vectors. In the research to date (See Chapter 2), the functions  $\underline{f}$ ,  $g_f$ ,  $g$  are usually such that pure optimal strategy functions  $\underline{u}^0(t)$  and  $\underline{v}^0(t)$  exist, and the object has been to determine these functions and the value function  $w(\underline{z}(\tau), T, \tau)$

$$w(\underline{z}(\tau), T, \tau) = \underset{(\underline{u}(\tau), \underline{v}(\tau))}{\text{val}} J(\underline{z}(\tau); \underline{u}(\tau), \underline{v}(\tau); T, \tau) \quad (7.3)$$

In some cases it has even been possible to find optimal closed-loop, or feedback, strategies so that  $\underline{u}^0(t) = \underline{u}^0(\underline{z}(t), t)$  and  $\underline{v}^0(t) = \underline{v}^0(\underline{z}(t), t)$ . The usual technique has been to apply either a method of characteristics or a Hamilton-Jacobi-Bellman method. The latter method requires the solution of

$$-\frac{\partial}{\partial \tau} w(\underline{z}(\tau), T, \tau) = \underset{(\underline{u}(\tau), \underline{v}(\tau))}{\text{val}} (g(\underline{z}(\tau), \underline{u}(\tau), \underline{v}(\tau), \tau) + (\nabla_{\underline{z}(\tau)}^T w(\underline{z}(\tau), T, \tau)) f(\underline{z}(\tau), \underline{u}(\tau), \underline{v}(\tau), \tau)) \quad (7.4)$$

When pure strategy solutions do not exist, the problem becomes more difficult. For differential games even the precise definition of what is meant by a mixed strategy can be elusive, although it will in some sense be a cumulative probability distribution  $F(\underline{u}(t))$  [or  $G(\underline{v}(t))$ ] over all admissible control functions  $\underline{u}(t)$  [or  $\underline{v}(t)$ ]. We might think of a closed-loop mixed strategy for the maximizer as a c.d.f.  $F(\underline{u}|\underline{z}(\tau), \tau)$ , with a similar function  $G(\underline{v}|\underline{z}(\tau), \tau)$  for the minimizer, and then choose the control vectors of each time instant  $\tau$  by making random draws from the proper distribution.

Defining these concepts precisely and computing the optimal strategies is rife with philosophical and mathematical difficulties. The obvious step of applying the method of dual cones to the pre-Hamiltonian on the right-hand-side of (7.4) is not really obvious

in implementation and, as we shall see, does not even seem to necessarily lead to definitive results. An intuitively acceptable approach is to discretize the differential game by taking a partition  $\Pi$  of the time interval  $[\tau, T]$  and to agree to let the controls  $\underline{u}$  and  $\underline{v}$  be constants within an interval  $(t_i, t_{i+1})$  of the partition. The resulting multistage game is solvable, at least in principle, and its value  $w_{\Pi}(\underline{z}(\tau), T, \tau)$  and mixed strategies for each interval may be found. We then accept the limit  $w^*(\underline{z}(\tau), T, \tau)$  of  $w_{\Pi}(\underline{z}(\tau), T, \tau)$  as the size  $|\Pi|$  of the partition  $\Pi$  goes to zero as the value of the differential game, provided that the limit exists, and similarly take the optimal mixed strategy limits as suitable for the differential game.

Fleming [55] shows that if  $f$  and  $g$  are continuous and satisfy a Lipschitz condition in  $\underline{z}$  and if  $g_f$  satisfies a Lipschitz condition on every bounded set, then the limit  $w^*$  exists; he conjectures that  $w^*$  is indeed the value of the differential game. In a more restrictive theorem, but one applicable for our problem, Fleming [53] proves that if a function  $w(\underline{z}(\tau), T, \tau)$  satisfies (7.4) and is continuously differentiable in an open set containing the region of interest, then

- (a)  $w(\underline{z}(\tau), T, \tau) = \lim_{|\Pi| \rightarrow 0} w_{\Pi}(\underline{z}(\tau), T, \tau)$  uniformly (7.5)
- (b)  $w(\underline{z}(\tau), T, \tau)$  is the value of the differential game with initial condition  $\underline{z}(\tau)$  at time  $\tau$  and fixed terminal time  $T$ .

The latter statement holds in the sense of  $\epsilon$ -effective

closed-loop strategies, that is, strategies which are arbitrarily close discrete approximations of continuous strategies.

Given this exceedingly brief background, let us first solve a simple example using limits of discrete approximations and then consider the question of direct evaluation of (7.4) for that example.

## 7.2 A FORMAL EXAMPLE

A very simple example will help illustrate some of the points to be made. Let the dynamics equation be

$$\dot{z} = u + v \quad z(0) = z_0 \quad (7.6)$$

where  $z, u \in [0, 1], v \in [0, 1]$  are scalars, and let a payoff function be given as

$$J(\underline{z}(\tau), \underline{u}, \underline{v}, \tau) = (z(T))^2 \quad (7.7)$$

We seek the value and optimal closed-loop mixed strategies for this game.

If (7.6) is approximated by

$$z_{i+1} = z_i + \epsilon (u_i + v_i) \quad (7.8)$$

where  $\epsilon = (T - \tau)/N, \tau \in [0, T]$ , then we find that we have a game which is of the type considered in previous chapters. In fact, since  $w_{N+1}(z) = z^2$ ,

$$w_N(z_N) = \underset{\text{val}}{(u_N, v_N)} [1 \quad u_N \quad u_N^2] \begin{bmatrix} z_N^2 & 2\epsilon z_N & \epsilon^2 \\ 2\epsilon z_N & 2\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ v_N \\ v_N^2 \end{bmatrix} \quad (7.9)$$

Letting  $w'_N(z) = w_N(z)/\epsilon^2$  and  $x = z/\epsilon$  gives

$$w'_N(x) = \text{val}_{(u, v)} [1 \ u \ u^2] \begin{bmatrix} x^2 & 2x & 1 \\ 2x & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} \quad (7.10)$$

which is precisely the same as the intermediate problem of Example

6.1. When we use the results of that example, we find

$$w'_{N-i+1}(x) = \max \left[ (x+i)^2, \frac{1}{4} \right]$$

$$F_i^0(u|x) = \begin{cases} I_0(u) & x < -i - \frac{1}{2} \\ \frac{1}{2} I_0(u) + \frac{1}{2} I_1(u) & -i - \frac{1}{2} \leq x \leq -i + \frac{1}{2} \\ I_1(u) & x > -i + \frac{1}{2} \end{cases} \quad (7.11)$$

$$G_i^0(v|x) = \begin{cases} I_1(v) & x < i - \frac{1}{2} \\ I_{-i+\frac{1}{2}-x}(v) & -i - \frac{1}{2} \leq x \leq -i + \frac{1}{2} \\ I_0(v) & x > -i + \frac{1}{2} \end{cases}$$

This may also be written in terms of  $w$  and  $z$  as

$$w_{N-i+1}(z) = \max [z^2 + 2i\epsilon + i^2 \epsilon^2, \frac{1}{4}\epsilon^2]$$

$$F_i^0(u|z) = \begin{cases} I_0(u) & z < (-i - \frac{1}{2})\epsilon \\ \frac{1}{2}I_0(u) + \frac{1}{2}I_1(u) & (-i - \frac{1}{2})\epsilon \leq z \leq (-i + \frac{1}{2})\epsilon \\ I_1(u) & z > (-i + \frac{1}{2})\epsilon \end{cases} \quad (7.12)$$

$$G_i^0(v|z) = \begin{cases} I_1(v) & z < (-i - \frac{1}{2})\epsilon \\ I_{-1+\frac{1}{2}-\frac{z}{\epsilon}}(v) & (-i - \frac{1}{2})\epsilon \leq z \leq (-i + \frac{1}{2})\epsilon \\ I_0(v) & z > (-i + \frac{1}{2})\epsilon \end{cases}$$

Taking  $\epsilon = (T - \tau)/N$ , holding  $T$  and  $\tau$  fixed, and letting  $N \rightarrow \infty$ , gives formally, for  $i = N$

$$w(z, T, \tau) = (z + (T - \tau))^2$$

$$F^0(u(\tau)|z(\tau), \tau) = \begin{cases} I_0(u(\tau)) & z(\tau) < -T + \tau \\ \frac{1}{2}I_0(u(\tau)) + \frac{1}{2}I_1(u(\tau)) & z(\tau) = -T + \tau \\ I_1(u(\tau)) & z(\tau) > -T + \tau \end{cases} \quad (7.13)$$

$$G^0(v(\tau)|z(\tau), \tau) = \begin{cases} I_1(v(\tau)) & z(\tau) < -T + \tau \\ I_{\frac{1}{2}}(v(\tau)) & z(\tau) = -T + \tau \\ I_0(v(\tau)) & z(\tau) > -T + \tau \end{cases}$$

This gives the value of the game starting at time  $\tau = 0$  and position  $z_0$  as  $w(z_0, T, 0) = (z_0 + T)^2$ , and yields optimal closed-loop strategies for the players for each  $\tau \in [0, T)$ .

Substituting in (7.4), we find that for each  $\tau$

$$\begin{aligned}
 2(z + T - \tau) &= \underset{(u, v)}{\text{val}} [2(z + T - \tau)(u + v)] \\
 &= \int_0^1 \int_0^1 2(z + T - \tau)(u + v) dF^0(u|z) dG^0(v|z) \\
 &\equiv 2(z + T - \tau)
 \end{aligned} \tag{7.14}$$

Therefore, by Fleming's results [53] we indeed have a solution to the problem.

### 7.3 SOLUTIONS USING LIMITS OF DISCRETE APPROXIMATIONS

The example in Section 7.2 is provocative in that it leads us to conjecture as to which differential game problems may be solved in that same manner. Solving the problems exactly appears to require that the discrete approximations be analytically solvable using the partition size as a parameter, which in turn seems to mean that the discrete problems must be such that the value for each stage is a polynomial and the stage patterns are repetitive so that induction on the stage index is possible. These are clearly restrictive assumptions.

If only approximate solutions are sought or if the problem is such that limit patterns are easily recognizable, then a much broader spectrum of problems may be attacked. In principle,

if  $\underline{f}$ ,  $g_f$ , and  $g$  in (7.1) and (7.2) are polynomials, then the method of dual cones may be applied to any discrete approximation to the differential game and the results of Chapter 5 may be applied.

More particularly, this may be done for a sequence  $\{\Pi_1, \Pi_2, \dots, \Pi_M\}$  of partitions of the time interval  $[0, T]$ , with  $|\Pi_{i+1}| < |\Pi_i|$ . This will yield sequences of value functions  $\{w_{\Pi_i}(z_0, T, 0)\}$  and of corresponding mixed strategies, and an approximate solution to the differential game may be taken either as one of the discrete versions or as a "guessed" limit of the sequence.

There are two important difficulties with the approximate approach. First, the value function may not be a polynomial in the region of interest, so that further approximations are necessary. We remark that, as shown in Chapter 5, this is not a problem if open-loop strategies are sought. The second difficulty is one of dimensionality, for if  $|\Pi_i|$  is small, then a great many subintervals will require processing. This may overburden a digital computer regardless of whether open-loop or closed-loop strategies are sought.

#### 7.4 SOLUTIONS BY ANALYSIS OF THE PRE-HAMILTONIAN

It is tempting to try to solve (7.4) directly, without resorting to limiting operations. Unfortunately, it is necessary to be very careful while doing this for it amounts to operating "at the limit" in situations where the higher order terms may be essential.

To illustrate this, let us first return to our example. In particular, suppose that the value is known to us but we are seeking the optimal strategies. Then we seek distributions such that

$$\begin{aligned}
2(z + T - \tau) &= \underset{(u, v)}{\text{val}} [2(z + T - \tau)(u + v)] \\
&= \min_G \max_F \int_0^1 \int_0^1 2(z + T - \tau)(u + v) dF(u|z, \tau) dG(v|z, \tau)
\end{aligned}
\tag{7.15}$$

The optimal distributions are obviously those of (7.13) provided that  $(z + T - \tau) \neq 0$ . However, if  $(z + T - \tau) = 0$ , then (7.15) does not yield information concerning the strategies. Thus there are both philosophical and practical difficulties in attacking the pre-Hamiltonian.

The reason for the difficulty with the above example is easy to find, for (7.4) is a limit of the discrete form

$$\begin{aligned}
\frac{w(\underline{z}, T, \tau) - w(\underline{z}, T, \tau + \epsilon)}{\epsilon} &= \underset{(\underline{u}, \underline{v})}{\text{val}} [g(\underline{z}, \underline{u}, \underline{v}, \tau) + [\nabla_{\underline{z}} w]^T \underline{f} \\
&\quad + \epsilon \underline{f}^T \frac{\partial^2 w}{\partial \underline{z}^2} \underline{f} + \dots]
\end{aligned}
\tag{7.16}$$

Ordinarily the terms on the r. h. s. containing  $\epsilon$  are ignored, for it is claimed that they are dominated by the first two terms. However, in our example this is not the case.

More generally, in solving discrete approximations using the principle of optimality we deal with equations of the form

$$\begin{aligned}
w_{\Pi}(\underline{z}, T, \tau) &= \underset{(\underline{u}, \underline{v})}{\text{val}} [\epsilon g(\underline{z}, \underline{u}, \underline{v}, \tau) \\
&\quad + w_{\Pi}(\underline{z} + \epsilon \underline{f}(\underline{z}, \underline{u}, \underline{v}, \tau), T, \tau + \epsilon)]
\end{aligned}
\tag{7.17}$$

In applying the method of dual cones to (7.17),  $\underline{z}$  and  $\epsilon$  are simply parameters in the solution. We have already seen that as the parameter  $\underline{z}$  varies, the set  $S(A(\underline{z}), R, \alpha)$  moves relative to the dual cone  $P_S^*$  and may possibly come to or cross a boundary from one form of strategy to another. This is particularly likely if a coefficient within  $A(\underline{z})$  passes through zero. Since  $\epsilon$  may well appear in (7.17) in such a manner that a coefficient in  $A(\underline{z})$  will be zeroed if  $\epsilon = 0$ , it is likely the problem for  $\epsilon = 0$  will be different in nature from the problem for  $\epsilon > 0$ . It seems, therefore, that equation (7.4) is useful for sufficiency checks on candidate solutions but is of limited value for synthesis purposes.

## CHAPTER 8

### SUMMARY, CONCLUSIONS, AND FUTURE WORK

In this report a viable solution technique for a special class of dynamic games has been created. The necessarily theoretical flavor of the approach must not be allowed to obscure the following fundamental result:

Two-person zero-sum noise-free multistage polynomial games of fixed duration may always be reduced to separable static games if open-loop mixed strategies are sought, and may often be reduced to sequence of such games when closed-loop mixed strategies are desired. The separable static games may then be solved as mathematical programming problems.

Of particular significance in applications is the fact that the technique is amenable to straightforward intuitively-satisfying numerical approximation; in fact, the well-developed methods and algorithms of linear programming may be used. These results were obtained and extensively discussed in Chapters 4 and 5, and they were illustrated in the examples in Chapter 6.

The method of dual cones, then, has been extended to the point that it may now be effectively applied to some real problems. Nevertheless, much work remains to be done. Numerical approximations should receive detailed attention in order that solutions may be obtained efficiently and precisely, and nonlinear programming formulations should be investigated. The form of the

value function must be investigated further; both theoretical questions of algebraic form and practical questions of numerical approximation require answers. The convex sets involved in vector problems need analytical description, if possible. These and related questions should be the subjects of immediate research.

Broader extensions of the method of dual cones may also be possible. The need for further investigation of its relationship to differential games is obvious. For example, an interpretation in which the sets  $S(A, R, w)$  and  $P^*$  move smoothly in relation to each other as time varies, with the direction of motion depending on the dynamics of the game, can be visualized. Some of the questions raised in Chapter 7 also bear answering.

Research should also be performed on the extension of the method to stochastic games. Several approaches appear possible here. One of the most intriguing possibilities is to note that imperfect knowledge of the state may mean that the set  $S(A, R, \alpha)$  is "fuzzy." Using this picture, it may then be possible to find not only a value but the distribution of the payoff.

Less obvious possible extensions undoubtedly exist, for mathematical game theory is an extensive field with many unsolved problems.

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