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CRITICAL PATH METHOD: A REVIEW

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CRITICAL PATH METHOD: A REVIEW

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This paper brings together different aspects of the Critical Path Analysis exclusively in terms of the mathematical developments in this area during the last eight years. The survey is divided into seven sections. The first one deals with the four stages of development, each successive stage being more representative of the real situation. The next five sections give explicit reference to the different articles which contribute to the stages of development described in the first section. The articles are reviewed in terms of the network algebra, PERT assumptions, cost/time trade off, resource allocation (heuristic) and resource allocation (analytical). Finally, the seventh and last section of this paper provides a summary of ideas and possible directions of research, some of which are currently being undertaken.

SECTION I: STAGES OF DEVELOPMENT

1.1 FIRST STAGE

The origin of the network approach to project management can be traced back to two separate projects; one undertaken by the private sector and the other by the U. S. Government, the latter being very much larger in magnitude and in the degree of uncertainty than the others. Both groups advocated the use of a network depicting explicitly the relationship among various activities. This was indeed a significant change from the then existing practice of using barcharts. The smaller of the two projects related to the construction industry where, by virtue of extensive experience, the activities could be well defined at the planning stage and unique time values ascribed to each one. This type of network can be classified as deterministic and activity oriented. Kelley and Walker [64] had initiated this under the title "The Critical Path Method".

The method consists of drawing a network of the project portraying the interrelationship among the activities and assigning time values (d_{ij}) to each activity (i,j). Every activity is uniquely represented by a predessor event(i) and a successor event(j) such that $i<j$.

The time calculations involve a forward pass to obtain the earliest occurrence (E_j) of successive events using the formula

$$E_j = \underset{i}{\text{MAX}} \left\{ E_i + d_{ij} \right\}$$

until the end event(N) is reached. Assuming that the latest occurrence of the end event (L_N) is the same as its earliest occurrence (E_N), backward pass calculations are made to find the latest occurrence of each predessor event until the start event is reached using the following formula:

$$L_i = \underset{j}{\text{MIN}} \left\{ L_j - d_{ij} \right\}$$

Thus, for each event, a measure of slackness (S_i) is obtained by the difference ($L_i - E_i$). Correspondingly, the total float (TF_{ij}) in activities are calculated from the formula

$$TF_{ij} = L_j - E_i - d_{ij}$$

On this basis, the activities having zero total float are called critical and a chain of such activities from the start to the end of the network is called the critical path. Several types of float calculations may be made for various purposes. The critical path (or paths) provides for management by exception among other benefits derived from this approach.

<u>Example</u> (1.1) There are five activities A, B, C, D and E having the durations 5, 2, 2, 7 and 3 respectively.

A precedes C and D. B and C precedes E. This implies that A and B can commence simultaneously while the completion of both C and D terminates the project. The following network (Fig. 1.1) includes all calculations and the critical path contains activities A and D in sequence.

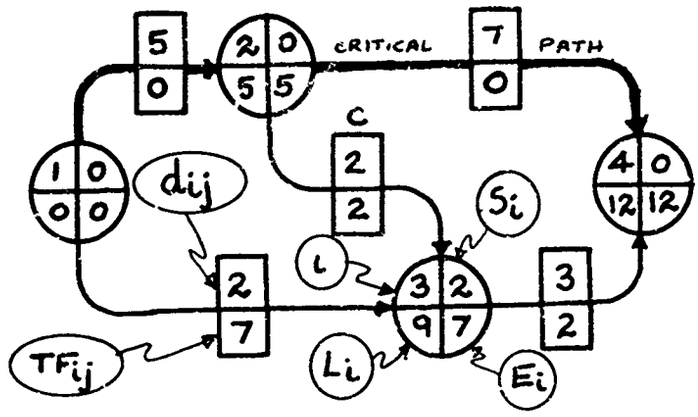


Fig. 1.1 Critical path calculations: deterministic case

In contrast to this, the group tackling a very large project relating to the Polaris Missile had based their system on Event Orientation. Since the project was highly complex and the first of its kind, it was not possible to determine the duration of individual activities precisely. The activity duration is assumed to follow a Beta distribution

$$dF = \left\{ \left[(b-a)^{1-p-q} \right] B(p,q) \right\} (t-a)^{p-1} (b-t)^{q-1} dt$$

and that the three time estimates t_1, t_2 and t_3 of a (optimistic), the mode, m_0 (most likely) and b (pessimistic), respectively, are available.

Appealing to the "empherical numerical manipulation" and truncating the normal distribution at ± 2.66 standard deviation, the formulaefor mean (d) and variance (V) aregiven by

$$d = \frac{1}{6}(t_1 + 4t_2 + t_3) \quad \text{and} \quad V = \left[\frac{1}{6}(t_3 - t_1)\right]^2$$

The expected value of the project duration is calculated by identifying the critical path as before using the expected activity durations. The distribution of the project duration is assumed to be normal with the mean value calculated as above. The variance at the project level is a simple addition of the activity variances along the critical path. (Mean = 5+7=12 ; Var. = (1/9)+(16/9) = (17/9))

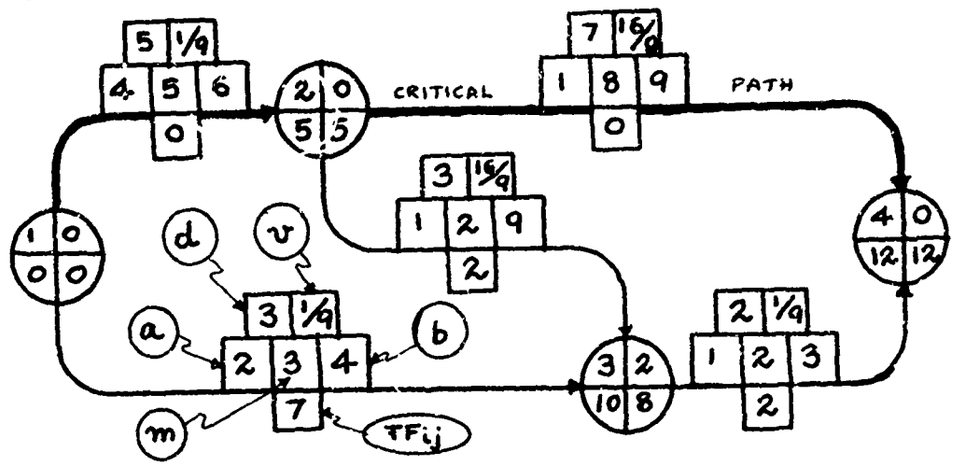


Fig. (1.2): Critical path calculations: Probabilistic case

A description of this approach is given under the title, Programme Evaluation and Review Technique in [80], the authors being Malcolm, Roseboom, Clark and Fazer. Several authors have examined the PERT assumptions rather

critically and discussions are still current as can be seen in Section Three. This type of network is therefore probabilistic and event oriented. These two papers, i.e., Kelly & Walker [64] and Malcolm, et al [80], mark the first stage of development in C.P.A. as given in exhibit [1].

1.2 SECOND STAGE

The exhibit [2] represents the second stage of development. At this stage, importance is given to the trade off between cost and time values of individual activities in such a way as to minimize total project costs. This section has been particularly attractive to operations research workers since, under varying assumptions, the problem may be approached in terms of one or more well known techniques of optimization. The two most important papers relating to this are due to Kelly [65] and Fulkerson [39]. These two papers assume that the cost vs. time distribution is bounded at both ends, continuous and convex. The network flow approach of Fulkerson also admits all-integer values for which an all-integer solution is obtained. Several authors have contributed to the extension of this basic model. A brief review of these papers may be seen in Section 4 of this paper.

1.3 THIRD STAGE

The second stage of development providing a time-cost trade off assumes an unlimited availability of various resources. Besides, the various elemental costs

DEVELOPMENTS IN CRITICAL PATH ANALYSIS

Exhibit I

STAGE I

Bar Charts / Net Work

Deterministic / Probabilistic

Event / Activity Orienctatior

Exhibit II

STAGE II

Time_____Cost

Continuous / Discrete / Mixed

Activity / Event on Node

Convex / Concave / Mixed

Bounded / UnBounded

Exhibit III

STAGE III

Time_____Cost

Time_____Resources

Divisible / Indivisible Activities

Decompose / Original Network

Unique / Alternative Deterministic Activities

Directed / Cyclical Network

Conjunctive / Disjunctive Activities

Single / Multiple Criteria

Single / Multiple Resources

Single / Multiple Network

pertaining to each activity are not indicated. The third stage of development includes the time/cost considerations in greater detail still independent of resource restrictions. The divisibility of activities, network decomposition, use of alternative paths and several other refinements are shown in exhibit [3]. The time/resource problem as outlined here is basically one of scheduling appropriate resources to different activities in such a way as to satisfy one or more objectives. Most of the published material on C.P.A. are included in these three exhibits, [1], [2] and [3], in one form or another. In particular, the exhibit [3] brings together recent publications in the four areas discussed in the next five sections of this paper: Network algebra, PERT assumptions, cost/time trade off and resource allocation (heuristic and analytical).

1.4 FOURTH STAGE

The exhibit [4] is an attempt to provide directions of research in C.P.A. The inter-connections shown in the exhibit represent only a few among several possible model refinements. For example, the problem of resource allocation has received considerable attention recently (see sections 5 and 6). Heuristic procedures have become sophisticated enough to include several realistic features of a project while the available analytical models are

Stage IV

RESEARCH DIRECTIONS IN CRITICAL PATH ANALYSIS

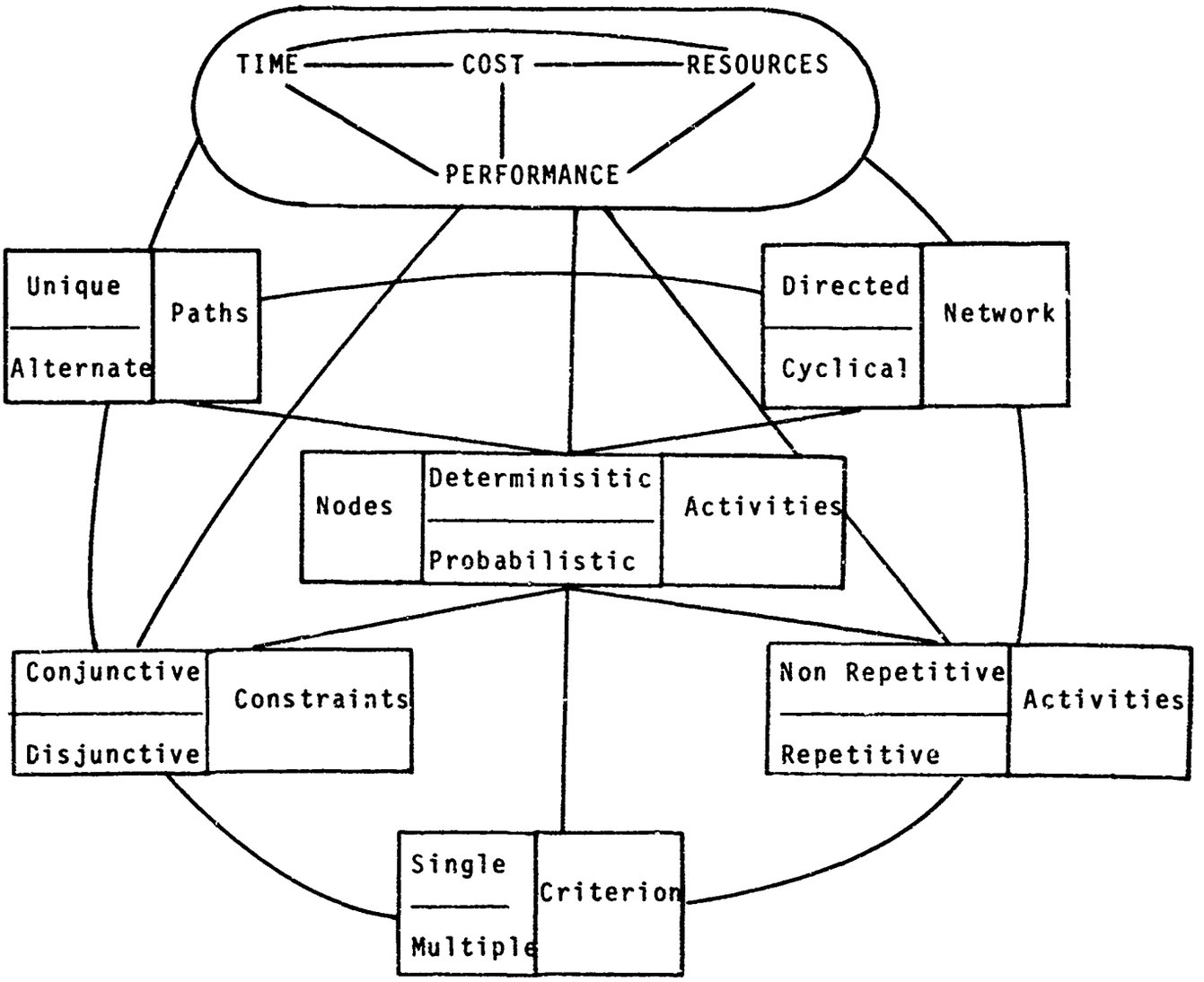


EXHIBIT IV

are unable to cope with large complicated projects. However, attempts are being made to combine costs and resources, using a heuristic as well as an analytical framework. The source material for the three stages of development .nd directions of research as depicted in exhibits [1], [2], [3] and [4] may be found in these sections. For example. "Divisible/Indivisible activities" referred to in exhibit [3] is derived from an article by Jewell [61] titled "Divisible Activities in Critical Path Analysis" as reported in section 2 and indicated in exhibit [5]. Similarily, "Conjunctive/Disjunctive activities" of exhibit [3] is related to the two papers by Roy and Sussmann [104] and Balas [2] which come under network algebra (exhibit [5]).

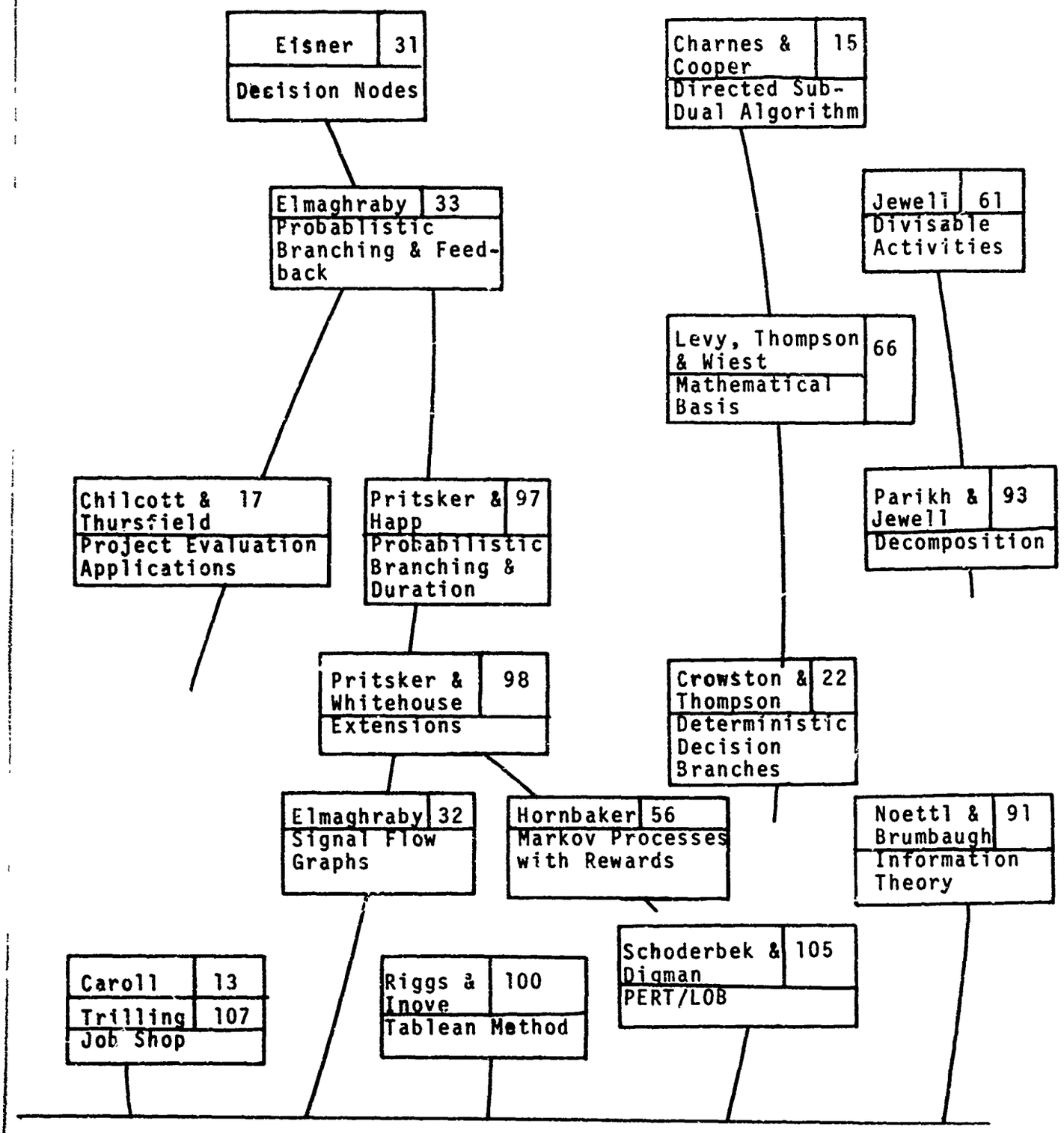
SECTION II: NETWORK ALGEBRA

2.1 AN OVERALL SURVEY

Irrespective of whether a network is event-oriented or activity-oriented, the project network as conceived by Kelley and Walker [64] and Malcolm, et al [80] is directed, acyclic with activities having deterministic or probabilistic durations. While conforming to this pattern, Charnes and Cooper [15] provide an interesting approach to the critical path calculations using what they call a "directed subdual algorithm." One of the early papers suggesting a variation from the basic pattern is due to Eisner [31] who introduced decision boxes in the network. Elmaghraby [33] further generalized this by introducing self loops and feedback loops. Chilcott and Thursfield [17] illustrate applications of these ideas in research management and project evaluation. Crowston and Thompson [22] introduce the title, "Decision C.P.M." for such networks containing decision nodes. Confining their attention to deterministic branching into activities, they provide algorithms employing time-cost trade off principles for reducing this type to a simple network having unique activities. Yet another change in the original network is in the use of disjunctive constraints due to Roy and Sussmann [104], further developed by Balas [2]. Even though these disjunctive

Exhibit V.

NETWORK ALGEBRA



constraints are currently used in simple machine-sequencing problems illustrated in a network form, the technique described by Balas has much wider potential applications in resource allocation. This section on Network Algebra (see exhibit 5) also includes contributions by Parikh and Jewell [93] and by Luttgen [78] on network decomposition, Fisher and Nemhauser [35] on multicycle project planning and Schoderbek and Digman [105] on Synthesis of PERT and Line of Balance. Even though these papers relate to the original directed acyctic network they have a significant bearing upon netowrk topology. The following paragraphs in this section cover some of these articles in greater detail. Further discussions on [2, 22, 35 and 93] are given in Section 4.

2.2 DIRECTED SUBDUAL ALGORITHM

Charnes and Cooper [15] convert a project graph, Fig. (1.1), to a flow network in which a hypothetical unit of flow (+1) leaves the start event 1 and enters the end event 4 . A flow of (-1) is ascribed to the end event. All intermediate events (u 2 and 3) play the role of transshipment points having conservation of flow. Accordingly, the activity duration, (d_{ij}), is interpreted as the time (or cost) of transporting a unit of flow from event(i) to event (j). The related linear

programming problem can now be formulated with the help of non-negative flow variables (x_{ij}) corresponding to activities (i,j), which maximizes the function $\sum_i \sum_j d_{ij} x_{ij}$ subject to the terminal event conditions $\sum_j x_{1j} = -\sum_i x_{iN} = 1$ and the flow conservation in transshipment points $\sum_i x_{ij} = \sum_k x_{jk}$ for each $j = 2, 3, \ldots\ldots N-1$. This generalized form assumes that $x_{ij} = 0$ if there is no activity connecting events (i) and (j).

With reference to Fig (1.1), the L.P. formulation for the primal and dual are as follows:

<u>Primal Problem</u>

Maximize $f(x) = 5x_{12} + 2x_{13} + 2x_{23} + 7x_{24} + 3x_{34}$

Subject to

$$
\begin{aligned}
x_{12} + x_{13} \qquad\qquad\qquad\qquad &= 1 \\
-x_{12} \qquad + x_{23} + x_{24} \qquad\qquad &= 0 \\
- x_{13} - x_{23} \qquad\qquad + x_{34} &= 0 \\
- x_{24} - x_{34} &= -1 \\
x_{ij} &\geqslant 0, \quad \text{all } (i,j)
\end{aligned}
$$

<u>Dual Problem</u>

Minimize $g(W) = w_1 - w_4$

Subject to

$$
\begin{aligned}
w_1 - w_2 \qquad\qquad &\geqslant 5 \\
w_1 \qquad - w_3 \qquad &\geqslant 2 \\
w_2 - w_3 \qquad &\geqslant 2 \\
w_2 \qquad - w_4 &\geqslant 7 \\
w_3 - w_4 &\geqslant 3
\end{aligned}
$$

Where

$$-\infty < w_i < \infty \quad , \quad i = 1, 2, 3, 4.$$

The dual form has the advantage of involving only two dual variables in each constraint (ie primal variable) and therefore by assigning a value to W_1, the absolute value of w_4 can be minimized simply by satisfying the dual constraints one by one.

Associated with each event (k), there is a dual variable (w_k) whose value is given by

$$\left| w_k^* \right| = \max_{\ell \in (i, \cdots j)} \left\{ d_{\ell k} + \left| w_\ell^* \right| \right\}$$

where the set ($i, \ldots j$) represents the set of all immediate predecessor events to event (k). Starting with ($w_1 = 0$), the values of (w_k), $k = 2, \ldots N$ can be obtained by repeated use of the above formula.

Accordingly,

$$w_2 \leqslant w_1 - 5 \quad , \qquad w_2^* = -5$$

$$\left.\begin{array}{l} w_3 \leqslant w_1 - 2 \\ w_3 \leqslant w_2 - 2 \end{array}\right\} \qquad w_3^* = -7$$

$$\left.\begin{array}{l} w_4 \leqslant w_2 - 7 \\ w_4 \leqslant w_3 - 3 \end{array}\right\} \qquad w_4^* = -12$$

In the usual critical path analysis, as given in Section (1.1). $(E_j - E_i) \geqslant d_{ij}$ and $(L_j - L_i) \geqslant d_{ij}$ and the critical path is identified by a sequence of activities (i, j) such that ($E_i = L_i$), ($E_j = L_j$) and

$(E_j - E_i = d_{ij})$. An analagous calculation of the dual L.P. problem would imply that for those dual constraints which are satisfied as equalities i.e. $w_i - w_j = d_{ij}$, e.g. first, third and fourth the corresponding primal variables x_{12}, x_{23} and x_{24} must be non-negative, where as it must be zero if the dual constraint is satisfied as an equality (eg: $x_{13} = x_{34} = 0$). Assigning a value 0 or 1 to the primal variable x_{ij} and checking the primal constraint equation ($x_{12} = 1$, $x_{23} = 0$, $x_{24} = 1$) it can be shown that $-w_N = \sum_i \sum_j d_{ij} x_{ij}$ given that $w_1 = 0$ where $|w_N^*|$ is interpreted as the minimum project duration

i.e. $-(-12) = (5 \times 1) + (2 \times 0) + (2 \times 0) + (7 \times 1) + (3 \times 0)$

The logic given above is contained in the directed subdual algorithm of Charnes and Cooper, who defines, for a network, a spanning tree which forms a connected graph with no loops. Accordingly every event will have one and only one activity of the project negatively incident on it. Furthermore, the spanning tree must have exactly one chain leading from the unique start event to the unique and event as in Fig.(2.2) and a flow of one unit passes through the links of this chain, while all flow variables corresponding to other activities have zero value. Then the chain of links bearing non zero flow variables is the critical path (i.e. x_{12} , x_{24}). Another illustration of this approach may be seen in Ref. [87], (pp. 136-139).

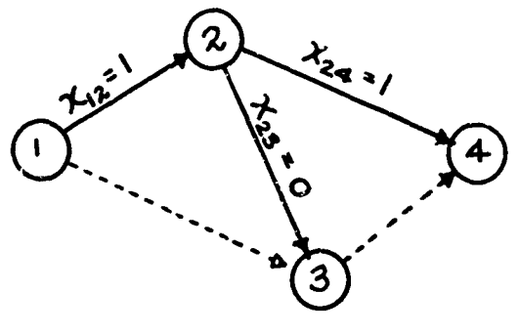


Fig. (2.2): The Spanning Tree

2.3 STOCHASTIC FEEDBACK NETWORKS

Elmaghraby's important paper [33] accounts for a significant departure from the well known directed acyclic networks having either deterministic or probabilistic activity durations. Based on the ideas of Eisner [31] regarding the probabilistic branching of activities from a given event (as is usually found in research and development where the outcome of an experiment determines further course of action), Elmaghraby evolves an algebra for generalized activity networks. Using the fundamental probability laws, he provides formulae for reducing activities in series, in parallal, in series-parallal, self loops and feed back loops into single activities.

The work of Elmaghraby [33] received further impetus as a result of three recent papers [97], [98] and [32], the last one being an added contribution by the same author. On the basis of an unpublished doctoral dissertation by Whitehouse [112] and a RAND memo by Pritsker [117], a new term was coined for the exploration of

stochastic activity networks: "G.E.R.T." to represent a Graphical Evaluation and Review Technique'.

With the help of Signal Flow Graphs [52, 57, 58, 76], Pritsker and Happ [97] have been able to outlive an algebra for the solution of networks having both probabilistic nodes and probabilistic activity duration. Elmaghraby's second article [32] also includes these extensions in network algebra. The formalization of node symbols, Fig.(2.31)

Node Characteristics and Symbols

Input / Output	*Exclusive-or*	*Inclusive-or*	*and*
Deterministic,			
Probabilistic,			

Exclusive-or—The realization of any branch leading into the node causes the node to be realized; however, one and only one of the branches leading into this node can be realized at a given time.

Inclusive-or—The realization of any branch leading into the node causes the node to be realized. The time of realization is the smallest of the completion times of the activities leading into the *Inclusive-or* node.

and—The node will be realized only if all the branches leading into the node are realized. The time of realization is the largest of the completion times of the activities leading into the *and* node.

Deterministic—All branches emanating from the node are taken if the node is realized, that is, all branches emanating from this node have a p-parameter equal to one.

Probabilistic—At most one branch emanating from the node is taken if the node is realized.

Fig. 2.31

and the fundamentals of network reduction for deterministic activity durations, Fig. (2.32) are given in these articles.

With regard to stochastic activity duration, Pritsker and Happ [97] consider the principles of network reduction for exclusive or nodes only: series, parallal, series-

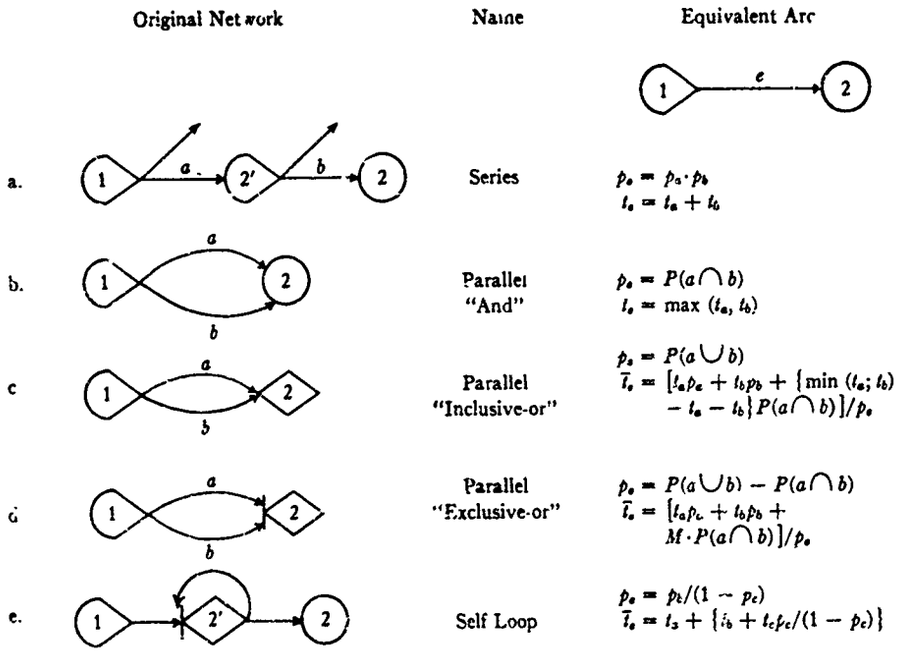
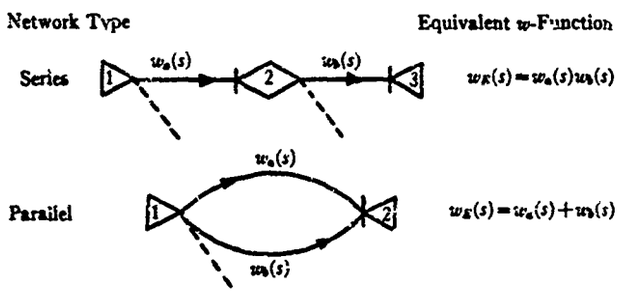


Fig. 2.32

parallal, and feedback loops. Essentially they employ a moment generating function (M.G.F.), $M_t(s)$, continuous or discrete, associated with activity duration (t) which is further transformed by the conditional probability (p) oriented to the branch to form a W-function $\omega(s) = p M_t(s)$. The outcome of this two stage transformation results in a system of linear independent equations which is amenable to flowgraph-techniques. The two key features of network reduction at the transform level are (i) The M-G-F of the sum of time values is the product of the M.G.F. of individual time values (ii) the M-G-F

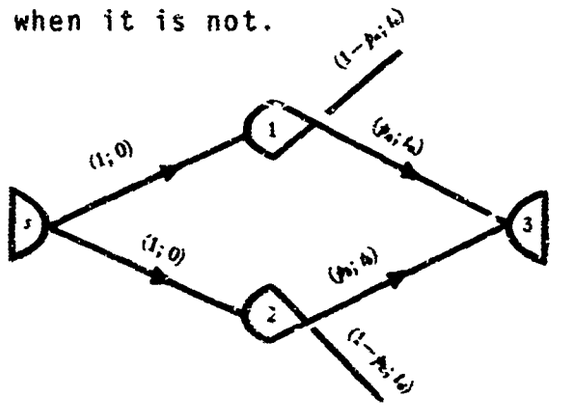
of a mixture of two distributions is the sum of M-G-F of individual distributions each one being weighted by their conditional probabilities. At the two stage transformation level this is reduced to simple addition of the corresponding functions as in Fig. (2.33). Complicated feedback networks are reduced with the help of Mason's Rule [118, 119].



Series and Parallel Networks.

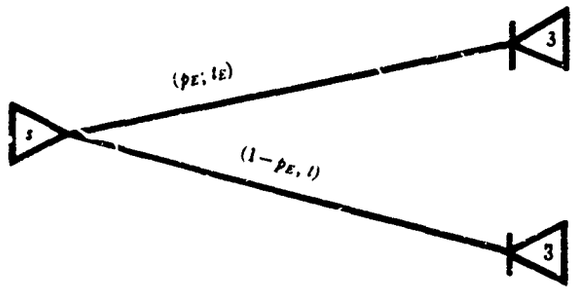
Fig. 2.33

Pritsker and Happ's article [97] also provides a systematic procedure for converting an 'inclusive-or' and 'and' type of nodes into equivalent 'exclusive-or' types given that the activity duration is deterministic. For example, an 'And' node, Fig. (2.34), can be replaced with two 'Exclusive-or' nodes, Fig. (2.35), one representing the eventuality when the 'And' node is realized, the other when it is not.



A Simple Network with an And Node.

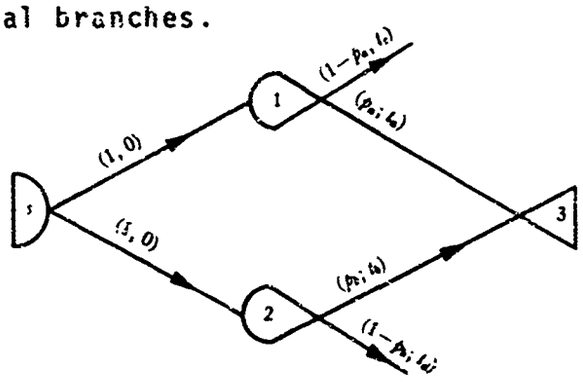
Fig. 2.34



Equivalent Network for an *And* Node.

Fig. 2.35

Similarly, an 'Inclusive-or' node, Fig. 2.36, may be replaced by a set of 'Exclusive-or' nodes. The original RAND memo of Pritsker [117] outlines possible approaches for problems of this type when the activity duration is probabilistic. Whitehouse [112] has investigated the use of tables, inversion integrals, Pearson curves and other approaches for obtaining the distribution function from an M-G-F or other exponential transforms so as to ascribe confidence limits to system performance. The authors of [97] conclude that further investigation is underway with regard to the sensitivity of the system performance measures to changes in parameters of individual branches.



A Simple Network with an *Inclusive-or* Node.

Fig. 2.36

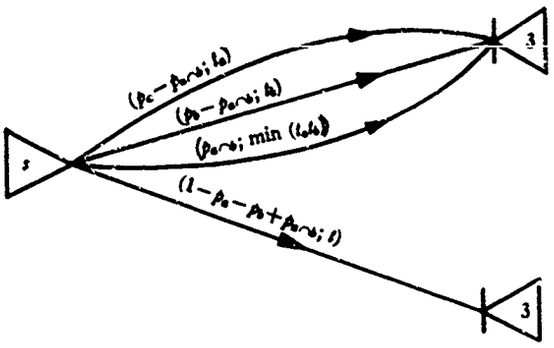


Fig. 2.37

Equivalent Network for an *Inclusive-or* Node.

In a companion article [98], Pritsker and Whitehouse present examples of multiple feedback loops, Multiple input and output nodes, selected problems in the probability theory, a model of a manufacturing process and finally a problem in research and development, originally conceived by Graham [46]. In terms of actual problems, GERT has been applied to the analysis of a space vehicle count down and the zone refining of semi-conductor material.

Hornbaker [56] applies GERT to the bond refunding decision problem originally defined by Bierman [5] as a Markov Process where the state of the process is described by an interest rate in each time period. Assuming that the firm breaks even refunding at the current bond interest rate, Bierman introduces a pay-off or reward matrix whose elements represent savings from refunding each interest rate below the break-even rate of interest for each time period. By converting the qualitative description of the bond refunding process to a model in a network form, Hornbaker applies GERT demonstrating the equivalence of results for the same data used in Bierman's

analysis. Even though, for this problem, Bierman's simple approach is a better one, GERT holds promise in the solution of such problems should they be made more complex.

2.4 JOBSHOP OPERATIONS

An entirely different point of view of network algebra is evident from the works of Caroll [13] and Trilling [107]. There are several job shop operations such as the building of large steam turbines which requires procedures of great complexity. Trilling [107] describes a coding procedure based on binany numbers while defining the networks represented by the routings or line ups of shop orders. The combinational routing shows the sequence in which various operations are performed on simple parts, combination of parts, assemblies and disassemblies. Whenever fabrication involves combinational routings, one basic consideration is how well coordinated the parts are that come together for assemblies. Delay in a part or a sub-group may cause a delay in the job order as in the critical path and in some cases aggravate the work-in-process inventory situation. Trilling considers several decision rules in job shop simulations. In particular, the works of Rowe [120] and Caroll [13] are significant.

FIG (2.51) TABLEAU METHOD

ES + ET = EF				A	B	C	D	E	LF - ET = LS			E - ET = B			I	IF	TF
0	5	5	A			/	/		5	5	0	5	5	0	0	0	0
0	2	2	B					/	9	2	7	7	2	5	0	5	7
5	2	7	C					/	9	2	7	7	2	5	5	0	2
5	7	12	D						12	7	5	12	7	5	5	0	0
7	3	10	E						12	3	9	12	3	9	9	0	2

NOTE : FOR EACH ACTIVITY (i,j), EQUIVALENT EVENT NOTATIONS FOR (i) OR (j) ARE GIVEN BELOW. FIG. 2.52 ILLUSTRATES THE SAME IN TIME SCALE.

ES = earliest starting time $\equiv$ E_i = earliest occurrence of event (i)

ET = expected duration $\equiv$ d_{ij}

EF = earliest finishing time = $E_i + d_{ij}$

LF = latest finishing time $\equiv$ L_j = latest occurrence of event (j)

LS = latest starting time = $L_j - d_{ij}$

$E = E_j$; $B = E_j - d_{ij}$; $I = L_i$

IF = independent float = $E_j - d_{ij} - L_i$

TF = total float = $L_j - d_{ij} - E_i$

2.5 TABLEAU METHOD

A recent paper by Riggs and Inoue [100] provides an interesting diversion from the main theme of network analysis. They suggest the use of a tableau having one row and one column for each activity. With this format, the network logic can be represented by filling the appropriate cells in this square tableau, leaving empty all the other cells as in Fig. (2.51). The various time calculations are then performed by extending this table, introducing more columns.

As is evident from Fig. (2.51), the tableau method eliminates the need for representing separately the events. The equivalence between event occurrence times and activity start or finish times is established in the note of Fig. (2.51) as well as in Fig. (2.52) which illustrates in the time scale, the values of total float and independent float for a given activity.

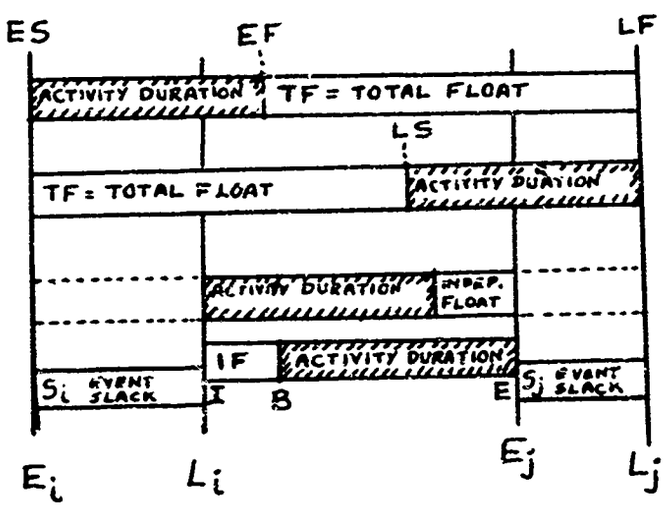


FIG. (2.52): FLOAT ANALYSIS

Riggs and Inoue also discuss its applications in cost-time trade off as well as resource allocation problems. They depend heavily upon the time scale diagram rather than the tableau to illustrate the computational features. However, the applicability of the tableau method in other areas is demonstrated by the classical "Stage Coach Problems" which is usually solved by dynamic programming.

2.6 DIVISIBILITY OF ACTIVITIES

Jewell [61] considers a project where an activity having a total time (U) may be devided and allocated with non-negative values (t_{ij}) to two or more locations (i,j) denoted by the subset D, $\{(i,j) \in D\}$, of the project network and for each i, ($i=1,2,\ldots\ldots N$), v_i represents its time of occurrence. The original primal, and its dual, as given in Fig. (2.61) are modified by bringing the primal constraint for (U) into the objective function with the help of an unknown LaGrange multiplier (Q). The modified form, Fig. (2.62), facilitates the use of a standard network flow approach. A parametric procedure is initiated with a large value of Q. As this value is decreased gradually, there is a resultant increase in the allocation of time (t_{ij}) to the D activities. The procedure terminates when this allocation reaches the desired level (U) <u>ie</u> $\sum\sum_D t_{ij} = U$

In the case of several divisible activities (K), the modified primal objective function becomes

$$\text{MIN} = (v_N - v_1) + \sum_{k=1}^{K} \left(\frac{1}{Q_k}\right)\left(U_k - \sum\sum_{D_k} t_{ij}\right)$$

The dual of this may be recognized as a multi commodity network flow problem. Jewell suggests further extensions of the model to include time-cost trade off principles. Specific procedures to achieve these extensions are still awaited.

PRIMAL	DUAL
$\text{MIN } T = v_N - v_1$	$\text{MAX } L = \left(\sum\sum_{\bar{D}}\right) T_{ij} Z_{ij} + Uy$
S.T.C.	S.T.C.
$v_j - v_i \geqslant T_{ij}, \left[(i,j) \in \bar{D}\right]$	$\sum_j \left(Z_{ji} - Z_{ij}\right) = \begin{cases} -1, & (i=1) \\ 0, & (i = 2,..N-1) \\ 1, & (i = N) \end{cases}$
$\left.\begin{array}{r} v_j - v_i - t_{ij} \geqslant 0 \\ \sum\sum_D t_{ij} \geqslant U \\ t_{ij} \geqslant 0 \end{array}\right\}, \left[(i,j) \in D\right]$	$Z_{ij} \geqslant 0, \left[(i,j) \in \bar{D}\right]$
	$-y + Z_{ij} \geqslant 0, \left[(i,j) \in D\right]$
	$y \geqslant 0$

FIG. (2·61)

MODIFIED PRIMAL	MODIFIED DUAL
$\text{MIN } T = Q(v_N - v_1) + \left(U - \sum\sum_D t_{ij}\right)$	$\text{MAX } \mathcal{L} = \sum\sum_{\bar{D}} T_{ij} X_{ij} + U$
S.T.C.	S.T.C.
$v_j - v_i \geqslant T_{ij}, \left[(i,j) \in \bar{D}\right]$	$\sum_j \left(X_{ji} - X_{ij}\right) = \begin{cases} -Q, & (i=1) \\ 0, & (i = 2,..N-1) \\ Q, & (i = N) \end{cases}$
$\left.\begin{array}{r} v_j - v_i - t_{ij} \geqslant 0 \\ t_{ij} \geqslant 0 \end{array}\right\}, \left[(i,j) \in D\right]$	$X_{ij} \geqslant 0 \left[(i,j) \in \bar{D}\right]$
	$X_{ij} \geqslant 1 \left[(i,j) \in D\right]$

FIG. (2·62)

SECTION III: PERT ASSUMPTIONS

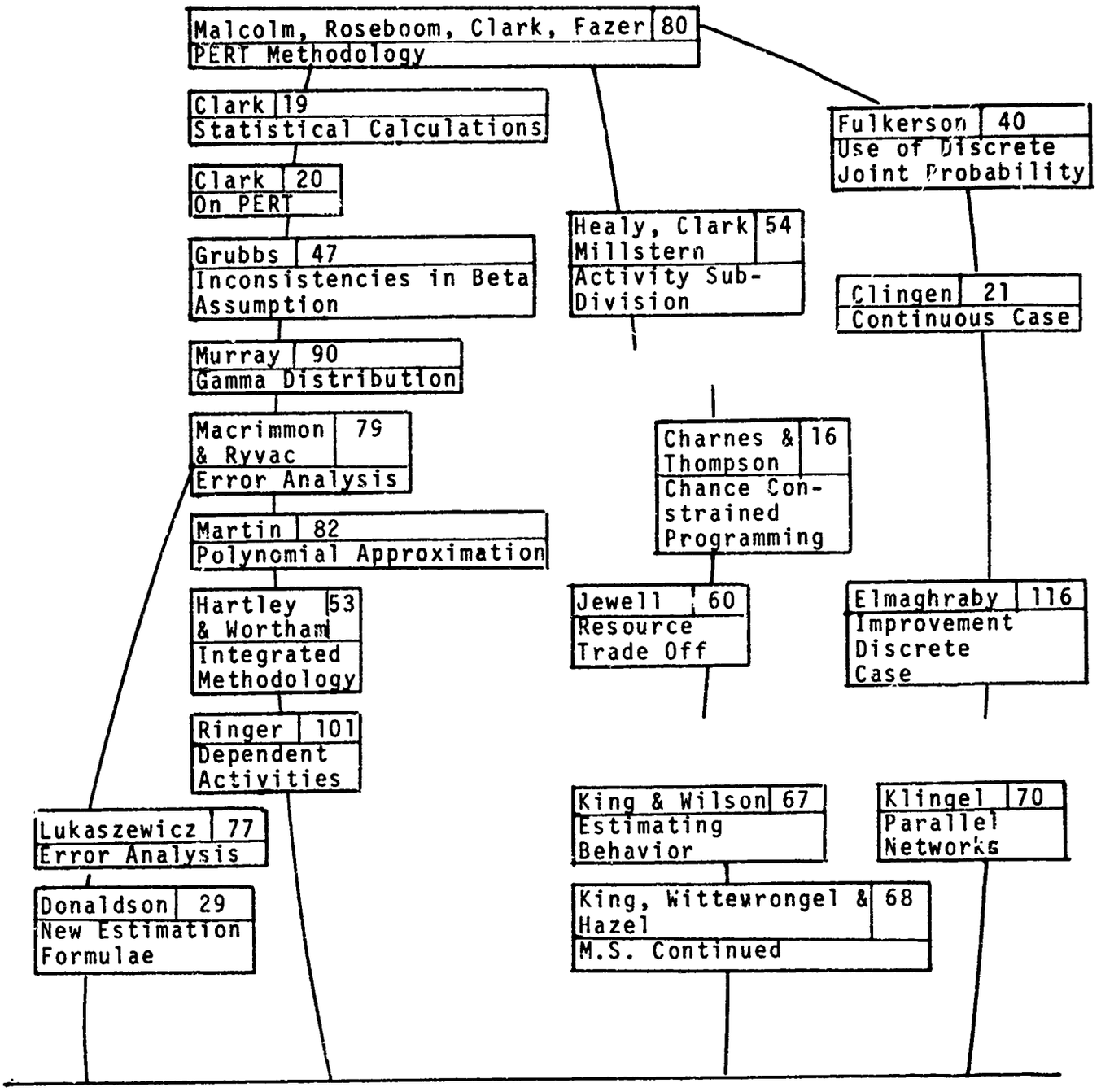
3.1 THE CLASSICAL CASE

This section is mainly devoted to directed acyclic networks whose activity durations are described by probabilistic distributions. The classical paper of Malcolm et al [80] assumes a Beta distribution for activity duration. The expected value and the variance of activity durations are derived from three easily observable points in the curve, the two ranges and the model value by means of two simple formulas as given in Section (1). The economic considerations of information gatherings and other attendant problems related to the complex, new and large project in their hand had led them to making several approximations both at the activity level and at the project level, ie the manner of aggragating individual activity time values through the network which lead to a probability distribution of the project duration. The authors then suggest that the probability of completing a project by a given date can be computed by calculating the critical path using and expected activity duration as deterministic quantities and then invoking the central limit theorem.

This approach is justified by the assumption that a critical path is comparatively longer than any other

Exhibit VI

PERT ASSUMPTIONS



path in the network so that the realization of a different path is least probable. Besides, the application of the central limit theorem is based on the observation that, in practice, the critical path would have enough number of activities to justify the same.

3.2 THE AGGRAGATION PROBLEM

While discussing the PERT assumptions, Murray [90] states:

> "This assumption states that the amount of time required to complete a network or to arrive at a given event within the network, has the same statistical distribution as the time required to complete the critical path leading into this event. Or, to state the assumptions more generally, the distribution of the maximum of a set of normally distributed random variables is equal to the distribution of the random variables with the larges first moment.
>
> This assumption was incorporated into PERT because it was needed to make possible the calculation of the expected completion times for various events in the PERT networks. This assumption was known to be invalid and to introduce a bias in the event completion date calculations

but it was thought that the biases were small. Work has been done towards ultimately correcting this assumption. Clark [19] has developed approximate methods for computing the moments of the maximum of a set of normally distributed random variables."

Given a simple network consisting of only two parallal activities having durations (t_1) and (t_2), means (μ_1) and (μ_2) and variances (σ_1^2) and (σ_2^2), Clark [19] shows that for the random variable that is the maximum of (t_1) and (t_2), the mean (μ_M) and the variance (σ_M^2) are given by

$$\mu_M = \mu_1 \phi(\alpha) + \mu_2 \phi(-\alpha) + a\, \theta(\alpha)$$

and

$$\sigma_M^2 = (\mu_1^2 + \sigma_1^2)\phi(\alpha) + (\mu_2^2 + \sigma_2^2)\phi(-\alpha) + (\mu_1 + \mu_2)\, a\, \theta(\alpha) - \mu_M^2$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

$$\theta(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$$

$$a = \sqrt{\sigma_1^2 + \sigma_2^2}$$

$$\alpha = \frac{\mu_1 - \mu_2}{a}$$

For the case of equal means , as in Tippet [106],

$\mu_1 = \mu_2$, $\alpha = 0$ and

$\mu_M = \mu_1 + \left(\frac{1}{\sqrt{2\pi}}\right)\sqrt{\sigma_1^2 + \sigma_2^2}$ and

$\sigma_M^2 = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)$

so even for this simple case, the mean and variance of the distribution of the maximum are dependent upon the variances of the two activities.

Clark [19] then assumes that the elements in the network are normal random variables, that the paths through the network are correlated, that the paths have unequal means and variances and that the distribution of the largest value is approximately normal. For such a case, he gives an iterative procedure whereby one integrates his way through the network eliminating an element at a time in terms of equivalent elements and finally ends up with the expected value and variances of the network. While summarizing Clark's bias correction procedure as given above, Klingel [70] observes that his approach is very tedious and also an approximation since it is necessary to assume normality at each iteration. Besides, the error of the approximation has not yet been clearly established. Moder and Phillips [87] (pp. 229-239) provide an illustrative application of this procedure.

3.3 COEFFICIENT OF SKEW

In support of the original paper by Malcolm et al [80], Clark, one of the authors of this paper, provides a practical explanation in [20], besides an attempt to validate the probability statements in terms of a truncated normal distribution as an appropriate simple model for specifying the ratio of the standard deviation to the range. However, Grubbs [47] has pointed out the subjective nature of the PERT estimation problem and the restrictions on the Beta Distributions that are required to validate the use of PERT means and variances.

Referring to the PERT formulae given in the first section-Grubbs shows that the parameters (p) and (q) of Beta distribution are restricted to the values.

$$p = 3 \pm \sqrt{2} \quad , \quad q = 3 \mp \sqrt{2} \quad \text{or} \quad p = q = 4$$

In particular, he indicates the existence of a coefficient of skew $[\gamma_1]$ given by the following equation.

$$\gamma_1 = \mu_3/\sigma^3 = 2(q-p)(p+q+1)^{1/2} \Big/ (p+q+2)(pq)^{1/2}$$

This is indeed inconsistant with the statement by the original authors that no assumption is made about the position of mode relative to the extreme values. Donaldson [29] suggests the use of the mean ($m_0 = t_2$) rather than the mode (m) in addition to the two limits of the distribution ($a = t_1$ and $b = t_3$) as the initial estimates. Accordingly $m_0 = (pt_3 + qt_1)/(p+q)$ and V' (variance estimator) $= (t_3 - t_2)(t_2 - t_1)/(p+q+1)$.

Assuming that $p > 2$ and $q > 2$, the smallest value of

($p+q$) which satisfies the necessary conditions is $(2+2K+\delta)$, where δ is a small positive value and

$$K = \max\left[(t_3-t_2)/(t_2-t_1) \quad , \quad (t_2-t_1)/(t_3 \quad)\right]$$

As pointed out by Coon, a referee of this paper, the family of curves generated by this approach do not lead to complete generalization of PERT activity time distributions, even though there is a considerable relaxation of the current restrictions.

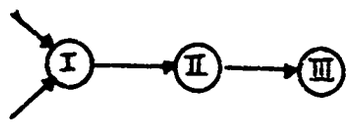


FIG.(3·41): INITIAL NETWORK

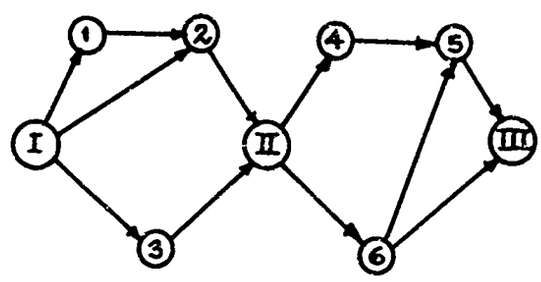


FIG.(3·42): SUBDIVIDED NETWORK

3.4 BIASES DUE TO ACTIVITY SUBDIVISION

In another early paper on this topic, Healy [54] investigates the bias introduced in the probability statements regarding project completion if the network activities are subdivided as from Fig. (3·41) to Fig (3·42). The completed probabilities vary depending upon whether the scheduled date for an event is later or earlier in time than the expected date. An important reason for this, as Healy points out, is that the standard deviation of the sum of two random variables (square root of the sum of two variances) is not the same as the standard deviations of the combined distribution. If the basis of the calculation as in PERT is based upon the approximation that for unimodal distributions, standard deviations can be estimated roughly as one ninth of the range.

Even though this represents an interesting mathematical excursion into probability calculations, Clark [54], a referee of this paper, comments that the decision to subdivide an activity is determined by the nature of the activity and that the decision is not a matter of analysis. Admittedly, it is not always easy to determine the appropriate amount of subdivision. Clark argues that this determination should be referred to the information source, not the analyst. Millstein [54], another

referee, comments that these probability estimates should be compiled with other factors such as the variances for expected event time, latest event time, schedule event time, the amount of event slack and the density of activities leading to the event. He further observes that in normal operating applications of PERT, an increase in the density of activities from 3 to 6 times the original planned schedule is abnormal.

3.5 JOINT DISTRIBUTION OF IMMEDIATE PREDECESSOR ACTIVITIES

Another important early contribution to PERT assumptions is due to Fulkerson [40]. One of his significant observations in this direction is that the PERT estimate (g) (ie use of expected activity duration in the deterministic calculation of the critical path for project duration) grossly underestimates the true expected project duration (e) which should include the whole activity distribution rather than its expected value. Obviously, it would be impossible to calculate the true mean for large networks having continuous activity distributions. However, Fulkerson suggests a method of obtaining a fairly good lower approximation (f) for expected project duration for discrete activity distribution such that $g \leq f \leq e$. The basic assumptions in his model are that the jobs which immediately precede any given job may have a joint distribution of times

$p(t_{B_i})$ while there is independence between jobs which immediately precede different jobs. For each event i, (given $i<j$) there is a bundle of all preceding activities (B_i), such that a set of activity durations $t_{B_i} = (t_1, t_2 \cdots \cdots t_{i-1})$ can be chosen with the appropriate joint probability distribution $p(t_{B_i})$ on activity duration within a bundle. Even though all preceding activities are included for generality, only the immediate predecessor activities and the respective predecessor events to event (i) are used in the calculations of the expected occurence (f_i) of event (i) (ie length of the path from origin to event (i)). Fulkerson defines these numbers (f_i) recursively as follows:

$$f_1 = 0$$

$$\vdots$$

$$f_i = \sum_{t_{B_i}} p(t_{B_i}) \max\left(f_1+t_1, \ldots\ldots, f_{i-1}+t_{i-1}\right)$$

He, then, illustrates the computational features through simple examples as given below, proceeding from one node to the next successor node taking into account all discrete probability values for each activity under consideration. When once the expected length of one node is calculated this is used as a basis for the next successor node. This process is comparable to the deterministic calculation of the critical path. This approach often results in an improved estimate even though more calculations are involved.

<u>Example 3.51</u>: Let the network be that of Fig. (3.51) and suppose that each arc independently assumes the length 0 or 1 with probability 1/2 (Ref.[40], p. 812)

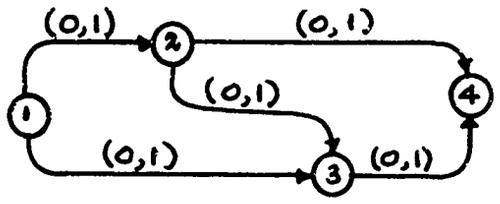


Fig. 3.51

The values of (g_i), $i=1,2,3,4$ are obtained by using the expected activity duration ($d_{ij} = 1/2$; all (i,j)) <u>ie</u>

$g_1 = 0$; $g_2 = g_1 + \frac{1}{2} = \frac{1}{2}$; $g_3 = \max\left(g_1 + \frac{1}{2}, g_2 + \frac{1}{2}\right) = 1$

and $g_4 = \max\left(g_2 + \frac{1}{2}, g_3 + \frac{1}{2}\right) = \frac{3}{2}$

The values of (f_i), $i=1,2,3,4$ are obtained by using the recursive formula given above.

$$f_1 = 0$$

$$f_2 = \frac{1}{2}(f_1 + 0) + \frac{1}{2}(f_1 + 1) = 1/2$$

$$f_3 = \frac{1}{4}\max(f_1+0, f_2+0) + \frac{1}{4}\max(f_1+0, f_2+1)$$
$$+ \frac{1}{4}\max(f_1+1, f_2+0) + \frac{1}{4}\max(f_1+1, f_2+1)$$
$$= \frac{1}{4}\left(\frac{1}{2} + \frac{3}{2} + 1 + \frac{3}{2}\right) = \frac{9}{8}$$

$$f_4 = \frac{1}{4}\max(f_2+0, f_3+0) + \frac{1}{4}\max(f_2+0, f_3+1)$$
$$+ \frac{1}{4}\max(f_2+1, f_3+0) + \frac{1}{4}\max(f_2+1, f_3+1)$$
$$= \frac{1}{4}\left(\frac{9}{8} + \frac{17}{8} + \frac{3}{2} + \frac{17}{8}\right) = \frac{55}{32}$$

Thus the statement " $g_i \leqslant f_i$, $i = 1, 2, 3$ and 4 " is validated by this example since $g = (0, 1/2, 1, 3/2)$ while $f = (0, 1/2, 9/8, 55/32)$. A formal proof for $g \leqslant f \leq e$ may be seen in [40].

Clingen [21] describes an extension of Fulkerson's algorithm to include the continuous case involving Stieltjes integration. The original formulation of the continuous case is not computationally feasible since in general it would involve multi-dimensional numerical integration. However, he derives an equivalent expression which for a discrete case is reduced to simple finite sums and for the continuous case, a one dimensional integration.

In continuation of the last two papers, Elmaghraby [116] suggests two methods both of which further improve upon Fulkerson's estimates of the true mean project duration. His first method is based upon the observation that even though the true mean of project duration is not affected whether the calculations are made in forward or backward directions, the intermediate events would have different values using Fulkerson's algorithm. Therefore, by obtaining the two estimates of expected length of a subnetwork from forward and backward calculations, the higher of the two values should be chosen for subsequent calculations. This would eventually lead to a better estimate of the project duration. The

second approach recommended by Elmghraby is a generalization of Fulkerson's procedure, by increasing the number of point estimates for each event and superimposing his first method of forward and backward probability calculations.

Elmaghraby's second approach would give a still better estimate of the true mean project duration. However, each improvement of original PERT calculations involves extra computation. Elmaghraby's article provides a basis for an effective trade off between computational cost and accuracy in estimating project duration, since his second approach can be generalized theoretically to any number of point estimates pertaining to each event, until the probability distribution function as a whole is taken into account.

3.6 INTERMEDIATE QUANTILES AND GAMMA DISTRIBUTION

Fulkerson [40], Clingen [21] and Elmaghrapby [116] as can be seen in Exhibit [6] form a stream of ideas brought up to date. On the other hand, Murray [90] is mainly concerned with the use of the Beta distribution, three time estimates and the approximate formula for the standard deviation. He shows the dependency of the formula for standard deviation upon the definition of the closed range, the optimistic and pessimistic values, and the unimodal beta distribution. Rather than having

defined boundaries, he suggests estimation of intermediate quantiles. He claims that this revision would more fully utilize the experience of the estimator and at the same time the derived estimates are more easily related to the moments of the distribution under consideration. Alternatively, he recommends the use of the Gamma distribution instead of the Beta distribution, since the former has the advantage of requiring only two time estimates to specify its parameter. However, the major defect of the Gamma distribution here is that its mode is always less than its mean. He has also advocated the use of probability papers in obtaining a rapid estimate of the bias introduced by PERT.

3.7 ERROR ANALYSIS AT ACTIVITY AND PROJECT LEVEL

MacCrimmon and Ryavac's paper [79] represents an important contribution in the systematic analysis of PERT assumptions. The bias problem is divided broadly into two categories: The activity level, the project level. At the level of individual activities they consider three possible sources of error. (1) The Beta distribution (2) the standard deviation and mean (3) imprecise three time estimates. They derive analytical expressions in each case to indicate the magnitude and direction of error. For example, the Beta distribution is compared with the quasi-uniform distribution at one extreme and

the quasi-delta function at the other extreme to obtain the bounds of error introduced in calculating the activity mean (error = $(1/3)(1-2m)$ and standard deviation ($1/6$). Commenting on this section of the paper, Lukaszewicz [77] suggests the use of a quasi-uniform distribution of a slightly different interval which improves upon the bounds of error. The second source of error due to simplified formulae for the standard deviation and mean are obtained by comparing these estimates with actual values of the standar deviation and mean expressed as a function of the mode and α, one of the parameters of the Beta distribution. The worst absolute error in the mean is

$$\left| \frac{1}{6}(4m+1) - m(\alpha+1) \Big/ (\alpha+2m) \right|$$

The worst absolute error in the standard deviation is

$$\left| \frac{1}{6} - \sqrt{m^2(\alpha+1)(\alpha-\alpha m+m) \Big/ (\alpha+2m)^2(\alpha+3m)} \right|$$

Similarly for the third case, the lack of precision in the estimating procedure is analyzed by allowing the three time estimates to vary within reasonable bounds, and the effect of this in the calculation of activity mean and standard deviation is studied. The worst absolute error in the mean is $\frac{1}{60}\left(\frac{a+4m+2b}{b-a}\right)$ while that of the standard deviation is $(1/30)\left\{\frac{b+a}{b-a}\right\}$.

The second part of MacCrimmon and Ryavac's paper is concerned with the calculations underlying the project

mean and standard deviation as well as the probability statements which follow these calculations. As observed by Fulkerson [40] earlier, the project duration is always biased optimistically by PERT calculations, while the standard deviation may be biased in either direction. They indicate that the network configuration influences the magnitude of errors at this level. In particular, the length of the critical path as compared to other paths, the number of sub-critical paths very close to the critical one, the number of common nodes among various paths, etc. tend to either compensate or aggravate the error values. However, it should be possible to eliminate certain activities or parallal paths from the network if the sum of the maximum time along this path is less than the sum of the minimum times along another existing path. In fact, this principle has been exploited by Handa [51] while considering time/cost trade off problems, so as to reduce the total computational effort.

3.8 USE OF MONTE CARLO METHOD

While MacCrimmon and Ryavac of Rand Corporation were working on a comprehensive analysis of PERT assumptions, Van Slyke [108], also of Rand Corporation, was exploring the use of Monte Carlo methods to yield solutions to the Pert problem. This method has a considerable flexibility in the use of any theoretical or empirical,

continuous or discrete distribution for activity duration. The effect of introducing or neglecting highly arbitrary assumptions relating to these distributions can be studied. As in other problems, the Monte Carlo approach here involves obtaining a sample value for each activity duration in the network and proceeding with the usual deterministic critical path calculations to obtain the project duration. A large set of trials and further statistical computations would yield distributions and parameters of interest. Either as a means of validating certain assumptions or as a basic method of computing desired results, Monte Carlo offers unique opportunities. Van Slyke observes that the Monte Carlo estimate of the mean project length is unbiased and the correlation between activities can be efficiently handled. Another important by-product of this approach is the availability of a 'criticality index' for each activity expressed in terms of the probability that the activity will be on the critical path. Van Slyke proposes two methods for computational short cuts, one of which has also appeared in MacCrimmon and Ryavac's paper, leading to the elimination of certain paths, if the activity duration has finite ranges. For other types of activities, Van Slyke suggests the elimination of all activities that were never critical within a limited number of trials.

3.9 HERE AND NOW FORMULATION

The article by Jewell [60] provides an interesting diversion from the type discussed in this section. He makes use of the principles involved in the cost/time trade off in the face of uncertain activity duration. At the outset, for each activity a preliminary level of effort measured in terms of money, manpower or other resources is determined and a probability distribution for this level of effort is given. With this format for each activity the planner estimates the interval needed for the completion of each activity. These estimates are then used in the usual way for the calculation of the project duration. The event times calculated at this state are fixed once and for all. In Jewell's "Here and Now" formulation of the problem, the activity duration becomes known only when it gets underway. Based upon the difference between the allotted time interval and the actual duration of the job, additional effort may be expended to stay on schedule. Assuming a convex, continuous time-resource curve having a quadratic form with defined boundaries, he employs the quadratic programming algorithm. This leads to a project resource curve where the minimal value of average additional effort expended is plotted against the project duration. The curves assume a piece wise quadratic form

3.10 POLYNOMIAL APPROXIMATIONS

An excellent theoretical contribution in PERT calculation is due to Martin [82]. His basic assumptions regarding the probability distribution of activity duration are that they are independent, have a finite range and are described by a class of polynomial density functions. The last assumption, which is the unique feature of this paper was chosen mainly to facilitate digital computation. The polynomial approximation does produce an error in the calculation of project duration, but Martin's approach provides a basis for a systematic study of network configuration. In this paper, he presents a detailed examination of the convolution operation for polynomials. By performing all computations in symbolic or algebraic form, the reduction algorithm described here circumvents the problem of trucation error usually found in numerical computation. His reduction algorithm converts a series-parallal network to an equivalent single activity. Since his algorithm also includes in a modified form the transformation of any directed acyclic network to series-parallal form, an efficient procedure is now available for finding the distribution of project completion time and the probability that an activity is on the critical path, "criticality index", as originally conceived by Van Slyke [108]. The main difficulty in the practical implementation of this approach

is in the computational requirements, in particular, the exponential growth of polynomial coefficients when the number of activities in the network increases [53]. The polynomial coefficients of the completion times for certain sub-networks are obtained from the recurrence formulae derived in his papers.

3.11 A NEW SYSTEM OF NETWORK CLASSIFICATION

Yet another attempt to remove the biases in PERT assumptions is made by Hartley and Wortham [53]. They present a new system of network classification which differentiates between 'uncrossed', 'crossed' and 'multiple crossed' networks. Their new definitions include terms such as an universal point, consecutive points, sets of second ordered branches and so on. This classification allows them to adopt different solution procedures for different types of networks depending on their degree of involvement and complexity. This method of analysis includes an operational calculus such as numerical integration for basic types of networks, a Monte Carlo procedure for more involved networks and analytical solutions for simple networks. The novel feature of this article is the attempt to synthesize various contributions in this direction and evolve a statistical theory for the derivation of unbiased distribution of the project completion times, with a provision for sensitivity analysis relating to assumptions.

3.12 REDUCTION ALGORITHMS

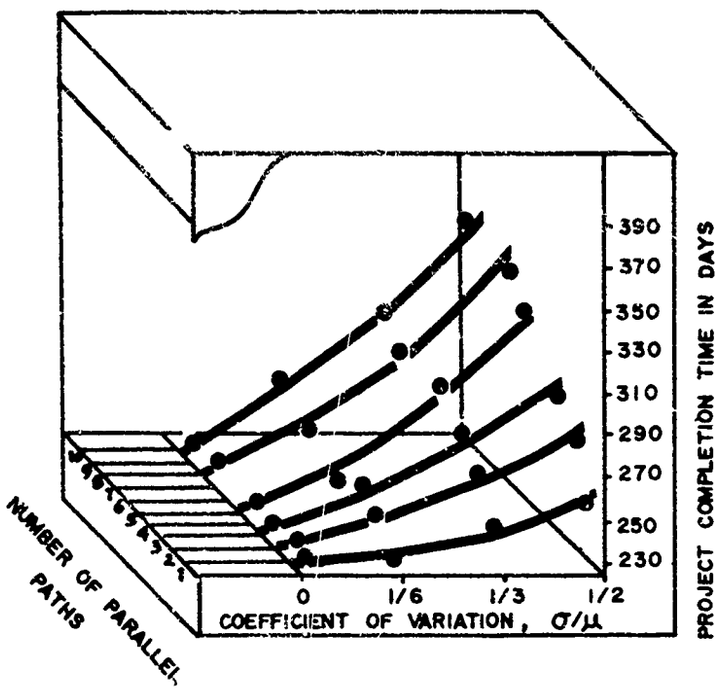
In his doctoral dissertation under Hartley, Ringer [101] presents algorithms to reduce the original PERT network to an equivalent network consisting of fewer activities. The distribution of completion times for K activities in series are reduced to a single distribution of the sum of K random variables. If the K activities are in parallal, the distribution of the maximum of K random variables is chosen. Certain types of crossed subnetworks can also be reduced. These algorithms are repetedly applied until no further reduction is possible either because a single activity remains or because crossed networks of a non reducible form remain. In the later case, Ringer outlines an efficient stratified sampling scheme as a part of the Monte Carlo technique to obtain the mean, variance and cumulative distribution function of the completion time. In addition to the above, he also presents a model which has dependency among activities in series or parallal.

3.13 EXPERIMENTS ON MULTIPLE PARALLAL PATHS

More recently, Klingel [70] reports an interesting study of a real network which indicates the bias introduced in project duration. The network chosen by him has multiple parallal paths with relatively equal means and large variances. Some of Klingel's observations regarding the works of Tippett and Clark in the analytical estimation of project duration have already been looked into in the

beginning of this section. His article gives details of experimental design and simulation procedures adopted for a large network of a thousand odd activities relating to multiple restaurant-service installations between 1 and 10. The network configuration is changed in terms of parallel paths, retaining the common constraints. With regard to the nature of activity distribution, Klingel has chosen Fisher's coefficient of variation (the ratio of standard deviation to the mean) as an appropriate measure of this variability. In all, 24 combinations between the number of parallel paths and the Fisher's coefficient of variation has been looked into, making 25 replications for each combination. The final result of this real example shows the incremental accumulation of bias in project duration up to 50%, as the variability in activity duration and the multiplicity of paths are increased as in Fig. (3.13). In Conclusion, Klingel makes an important observation pertaining to resource allocation in projects.

> "There are a large number of resource allocation programmes available which reduce deterministic completion time by increasing the parallelism of the network. While increasing parallelism reduces deterministic completion time, it also reduces the probability of completing on or before the new deterministic date."



Geometry of response surface

Fig. (3.13)

3.14 ESTIMATING BEHAVIOR OF INDIVIDUALS

The two most recent papers in this topic, King and Wilson [67] and King, Vittebrongel and Hazel [68] represent another interesting diversion from the basic theme. The second paper referred to above is more or less a continuation of the first one, validating the tentative conclusions reached earlier. This series fulfills a long felt need to initiate research on the estimating behavior of individuals in relation to PERT assumptions. The research group has collected a wealth of data about past subjective judgements of estimators in testing two hypotheses:

(1) Preactivity time estimating accuracy improves as the beginning date of the activity approaches.

(2) Time estimates made during the progress of an activity improve in relative accuracy as the completion date of the activity approaches.

Contrary to the general beliefs as stated in the two hypotheses, the researcher found that the intuitive learning concept inherent in first hypothesis was totally lacking in the pre-activity estimating behavior of individuals, even if they were given ample opportunities to re-estimate. The second paper reports the findings of five different government contracts. Strangely enough, even the second hypothesis could not be validated by their test results. Hence they conclude that there is no significant change in the accuracy of estimating the remaining portion of an activity as the portion of activity remaining becomes smaller.

On the basis of this information, the study group proposes the establishment of a super network which incorporates adjusted time estimates, under the assumption that there is a basic stability in the structure of the estimating system. King and Wilson consider several cost models in establishing the adjustment procedure and they conclude that models based on the minimization of total error cost may be useful in adjusting time estimates if valid cost relationships can be established.

3.15 CHANCE CONSTRAINED PROGRAMMING

The direct and the dual linear programming approach to the deterministic network time calculations are given in subsection (2.2) where Charnes and Cooper [15] present a directed sub dual algorithm. The problem may now be rephrased as follows:

Primal: $\max \sum_{j=1}^{n} t_j x_j$

SUBJECT TO

$$\sum_{j=1}^{n} \epsilon_{ij} x_j = a_i \quad ; \quad (x_j \geqslant 0) \quad (i = 0, 1, \ldots m)$$

where $a_0 = -1$, $a_n = 1$ and all other $a_i = 0$, t_j is the duration of activity (j) and ϵ_{ij} are the incidence numbers for the network.

Dual: min. $\sum_{i=0}^{m} u_i a_i$

SUBJECT TO $\sum_{i=0}^{m} u_i \epsilon_{ij} \geqslant t_j$

Charnes, Cooper and Thompson [16] present a stochastic version of this problem by assuming that the t_j values are random variables. Accordingly the dual problem consists of

MIN $\sum_{i=0}^{m} u_i a_i$, SUBJECT TO

$$P\left(\sum_{i=0}^{m} u_i \epsilon_{ij} \geqslant t_j\right) \geqslant \beta_j$$

where P means 'probability and $0 \leqslant \beta_j \leqslant 1$ is a preassigned

probability measure for $j = 1, 2, \cdots n$. This programming problem is therefore chance constrained. Let F_j be the marginal distribution of t_j such that

$$P\left(\sum_{i=0}^{m} u_i \epsilon_{ij} \geqslant t_j\right) = F_j\left(\sum_{i=0}^{m} u_i \epsilon_{ij}\right)$$

This leads to the inequality

$$\sum_{i=0}^{m} u_i \epsilon_{ij} \geqslant F_{ij}^{-1}(\beta_j) \quad , (j = 1, 2, \cdots n)$$

where $F_j^{-1}(\beta_j)$ represents the Fractile associated with β_j. On this basis, the primal has the form:

$$\text{MAX} \sum_{j=1}^{n} F_j^{-1}(\beta_j) x_j$$

$$\text{SUBJECT TO} \quad \sum_{j=1}^{n} \epsilon_{ij} x_j = a_i \quad , (x_j \geqslant 0), (i = 0, \cdots n)$$

Using this approach, Charnes, Cooper and Thompson [16] show the method determining the critical path for every possible collection of (t_j) realizations with their associated probabilities. Curves, based on electronic computation for a simple network with exponential activity durations are also given. The interacting properties of the network show that the overall behavior of the project duration is very different from that of individual activities, even for this monotonically decreasing curve. The use of this approach for other types of curves and several other extentions of the basic model are still under progress.

SECTION IV: COST/TIME TRADE OFF

4.1 THE PROBLEM

Let the duration (Y_{ij}) of each activity (i,j) vary between the crash time (a_{ij}) and the normal time (b_{ij}), (ie $a_{ij} \leqslant Y_{ij} \leqslant b_{ij}$) and let each unit decrement in time requires an additional cost (C_{ij}), the cost slope. The activity cost curve is therefore linear, continuous and bounded at both ends. This variability of duration at the activity level introduces several feasible schedules at the project level for each feasible project duration (λ).

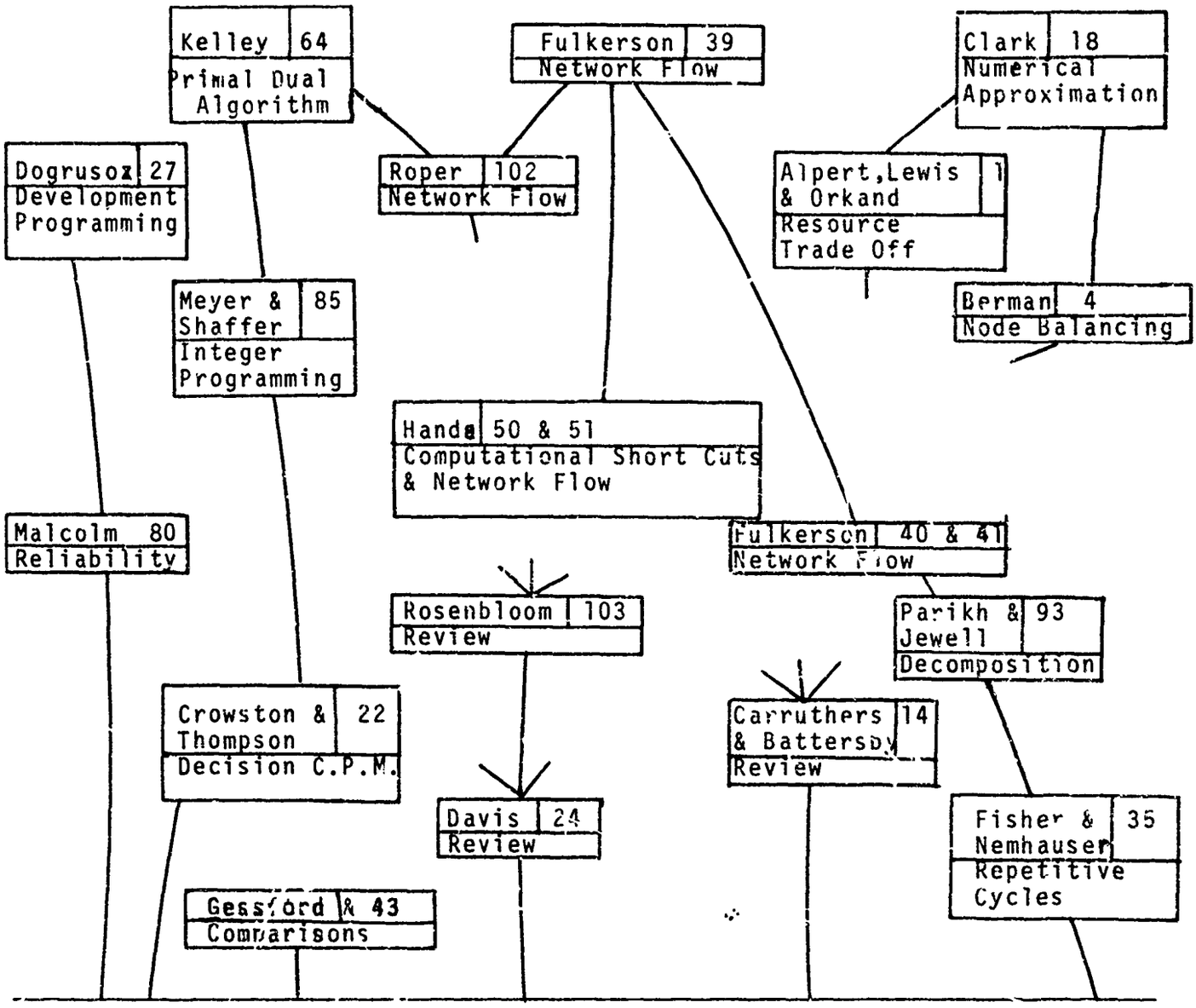
Associated with each schedule are the direct and indirect project costs which lead to total project cost curve. Several approaches are available in obtaining the direct project cost curve. This section is related to the computational details of this problem under varying assumptions.

4.2 USING THE CRITICAL PATH

A simple procedure involving the critical path calculations is given below. The example in subsection (1.1) is suitably modified to include cost slopes and crash durations as given in fig. (4.10).

Exhibit VII

COST / TIME TRADE OFF



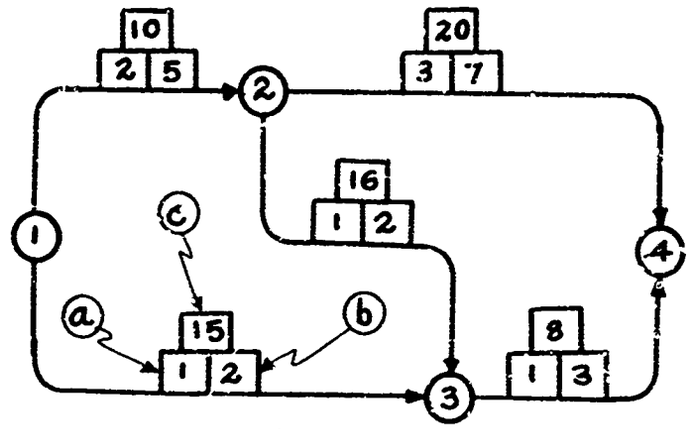


FIG. (4.10)

(i) For each activity, let $Y_{ij} = b_{ij}$ and compute the project duration (eg: 12 weeks) and identify the critical path (eg; 1, 2, 4).

(ii) Choose among the critical activities, the one with minimum cost slope [eg: min. (10, 20) = 10].

(iii) Compress that activity until:

(a) either $Y_{ij} = a_{ij}$ for that activity (eg: $Y_{12} = 2$)

(b) or some other path becomes critical (eg· at the end of the second iteration, Y_{24} is reduced from 7 to 5 weeks instead of 3 weeks) This situation leads to multiple critical paths[eg: (1,2,4) and (1,2,3,4) for a project duration of 9 weeks].

(iv) In the case of multiple critical paths, step (ii) requires the computation of joint cost slopes. [eg: at the end of the second iteration, the joint cost slope of (2,4) and (2,3) is 20 + 16 = 36 and that of (2,4) and (3,4) is 20 + 8 = 28]. Accordingly choose minimum among the set of cost slopes and joint cost slopes relating to the multiple critical paths (eg choose (2,4) & (3,4)).

(v) Proceed with the steps (ii) to (iv) until, for each activity of at least one critical path, $Y_{ij} = a_{ij}$ and therefore, no more compression is possible [eg: by compressing both (2,4) and (3,4) simultaneously by 2 weeks at the third iteration, the activities (1,2) (2,4) are now set at their crash times (eg 2 weeks and 3 weeks) and this terminates the compression].

The direct project cost curve for this example is shown in Fig. (4.11). The curve is piece-wise linear. By adding to this an indirect cost curve (having a positive cost slope) a convex total cost curve is obtained.

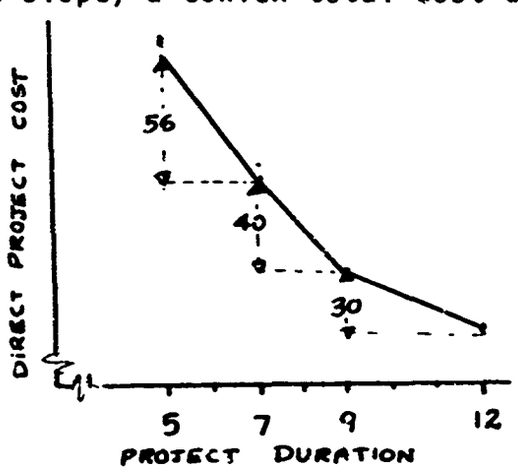


FIG. 4.11 DIRECT PROJECT COST CURVE

4.3 NETWORK FLOW APPROACH

Kelley [65] formulates the problem given above as a parametric linear programming problem. The objective function consists of a maximization of the project utility which is a function of the project duration. The constraint set has precedence and terminal restrictions of the network in addition to the upper and lower bounds on activity duration. Kelley develops a fairly efficient algorithm by incorporating a slight variation of Ford-Fulkerson's Network Flow algorithm [36].

In a companion paper by Fulkerson [39], a similar network flow approach is employed to minimize the total project cost. A brief summary of the primal, dual and modified dual formulation due to Fulkerson is given in Fig. (4.31). As illustrated by Elmaghraby [121], pp. (113-129), Fulkerson's flow algorithm may be divided into five steps. Using the same example given in the last section, details of repeated computation of these steps are given in Fig. (4.32). Even though, the modified dual form in Fig. (4.31) indicates the need for two parallel arcs for each (i,j), only one of them having the capacity C_{ij} is exhibited in Fig. (4.32) to maintain simplicity of exposition. The second arc having infinite capacity is added as and when necessary.

<u>Step (1)</u>: For each (i,j) and ($k=1,2$), set $f_{ij}^{(k)}=0$ and $Y_{ij}=b_{ij}$; compute maximum project duration ($\bar{\lambda}$), and

PRIMAL	DUAL	MODIFIED DUAL
MAX $\sum_{(i,j)} C_{ij} Y_{ij}$ S.T.C. $t_i + Y_{ij} - t_j \leq 0$, all (i,j) $-t_1 \quad + t_n \leq \lambda$ $Y_{ij} \leq b_{ij}$, all (i,j) $-Y_{ij} \leq -a_{ij}$, all (i,j) $C_{ij} \geqslant 0$, $b_{ij} \geqslant 0$, $a_{ij} \geqslant 0$ for all (i,j) activities, where c - cost slope a - crash time b - normal time λ - project duration t_i - occurrence time: event (i).	MIN $\lambda v + \sum_{(i,j)} b_{ij} g_{ij} - \sum_{(i,j)} a_{ij} h_{ij}$ S.T.C. $f_{ij} + g_{ij} - h_{ij} = C_{ij}$, all (i,j) and $\sum_j (f_{ij} - f_{ji}) = \begin{cases} v & ,(i=1) \\ 0 & ,(i=2,\dots n-1) \\ -v & ,(i=n) \end{cases}$ where f, v, g and h are the respective dual variables corresponding to the four rows of primal constraints. Since g_{ij} or h_{ij} must be zero at the optimum, the first equality constraint becomes $g_{ij} = \max[0, C_{ij} - f_{ij}]$ $h_{ij} = \max[0, f_{ij} - C_{ij}]$ and the network flow interpretation g - residual capacity, which is linear in (f) $0 \leq f \leq C$ h - reverse capacity, which is linear in (f) $C \leq f \leq \infty$ f - flow in the arc C - arc capacity, a constant.	MIN $\lambda v - \sum_{(i,j)} b_{ij} f_{ij}^{(1)} - \sum_{(i,j)} a_{ij} f_{ij}^{(2)}$ S.T.C. $\sum_{j,k} \left(f_{ij}^{(k)} - f_{ji}^{(k)} \right) = \begin{cases} v & ,(i=1) \\ 0 & ,(i=2,\dots n-1) \\ -v & ,(i=n) \end{cases}$ $f_{ij}^{(1)} \leq C_{ij}$, all (i,j) and $f_{ij}^{(k)}, v \geqslant 0$, $k=1,2$. This dual with piece-wise linear objective function has the following interpretation. In a network of (n) nodes, two arcs now represent the single arc (i,j). The first of these two has a capacity (C_{ij}) while the second has (∞). The problem is to construct a flow $f_{ij}^{(k)}$ from node (1) to node (n) in the new network which minimizes the above objective function subject to the "flow conservation" constraints.

FIG. (4.31) NETWORK FLOW MODEL FOR TIME-COST TRADE OFF.

assign label ($f_{ij}^{(1)}$, $C_{ij} - f_{ij}^{(1)}$) or ($f_{ij}^{(2)}$, ∞), the latter one for the infinite capacity arc.

Step (2): Label the origin, (ie node 1) with ($\infty, 0$). Check each arc originating from node 1 for the relationship $S_{ij}^{(2)} = a_{ij} - t_j = 0$, where t_j is the earliest occurrence of event(j). If so, label node (j) with(∞, 1). Proceed to label successive nodes with (∞, i), if the relationship $S_{ij}^{(2)} = t_i + a_{ij} - t_j = 0$ holds good. If the terminal node (N) can be labeled ($\infty, -$), the algorithm terminates. Otherwise, follow the subsequent analysis, maintaining the node labels (∞).

Step (3): Again, start with the origin having the label (∞, 0). For any labeled node (i), check all nodes (j) connected with (i) for the following relationship $S_{ij}^{(1)} = t_i + b_{ij} - t_j = 0$ and $f_{ij} < C_{ij}$. If so, label node (j) with (q_j , i) where $q_j = \min\left(q_i ; C_{ij} - f_{ij}^{(1)}\right)$ Should the relationship $S_{ij}^{(2)} = t_i + a_{ij} - t_j = 0$ hold good, label node (j) with (q_i , i). Continue labeling. If terminal N is labeled, a break through has occurred and so proceed to step (4). Otherwise, proceed to step (5).

Step (4): Start with node (N). For any labelled node (j) with label (q_j , i), increase the flow (f_{ij}) by (q_N), reduce the residual capacity in arc (i,j) by (q_N) and erase the labeling of (j). Continue this process, until node (1) is reached and go back to step (2).

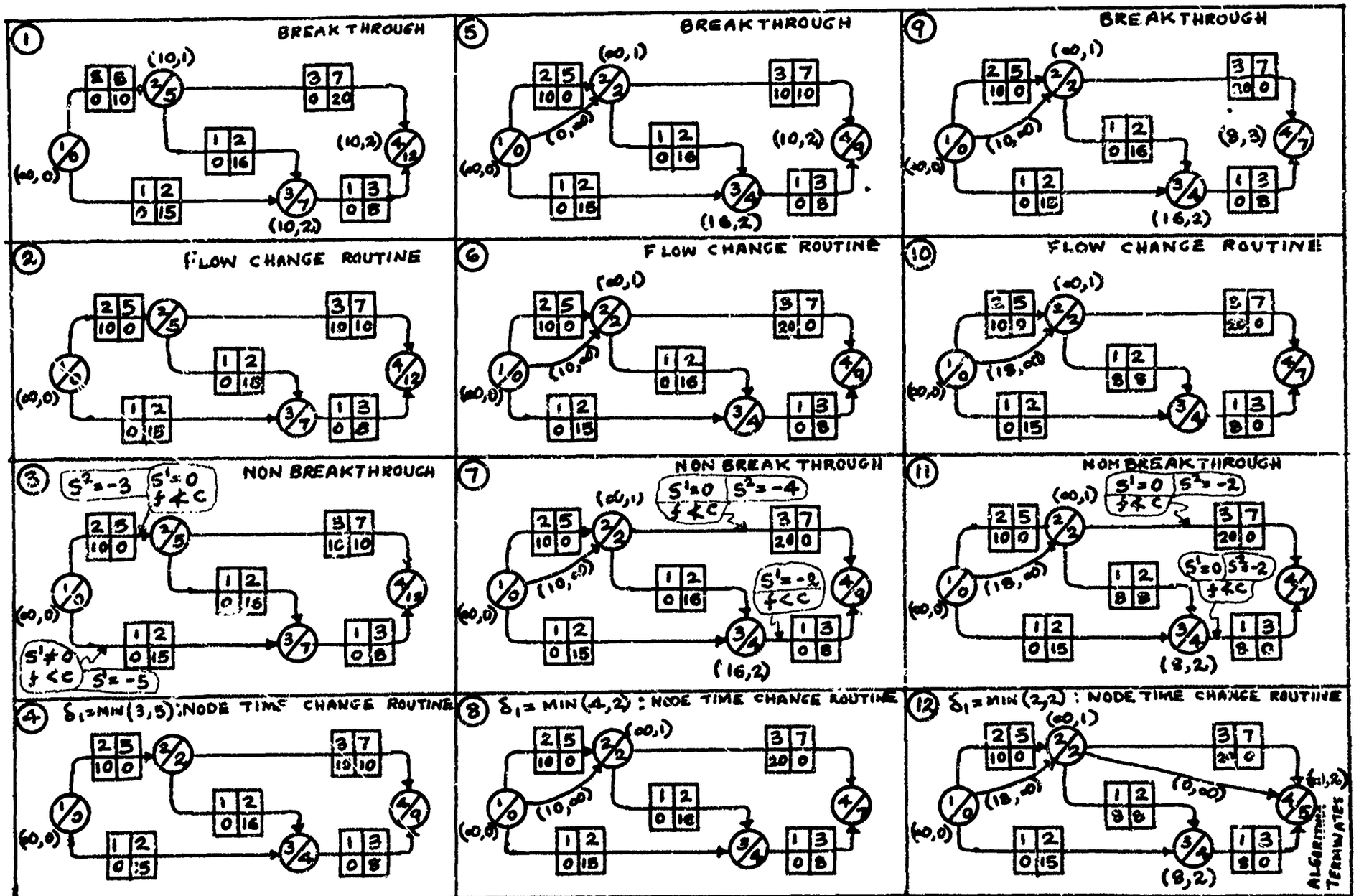


FIG. (4.32) ILLUSTRATION OF FULKERSON'S NETWORK FLOW ALGORITHM

Step (5): Define a "cut set" $Z = [Z_1, Z_2]$ of arcs $\{(i,j)\}$ such that one node is labelled and the other is not labelled, where, for $k = 1, 2$,

$$Z_1 = \{ (i,j) \mid (i) \text{ is labeled}, (j) \text{ is not labeled}, S_{ij}^{(k)} < 0 \}$$

$$Z_2 = \{ (i,j) \mid (i) \text{ is unlabeled}, (j) \text{ is labeled}, S_{ij}^{(k)} > 0 \}$$

Ignore all arcs of Z_1 where the equality $S_{ij}^{(k)} = 0$ holds.

Define
$$\delta_1 = \min_{Z_1} \left[- S_{ij}^{(k)} \right], \quad S_{ij}^{(k)} < 0$$
$$\delta_2 = \min_{Z_2} \left[+ S_{ij}^{(k)} \right], \quad S_{ij}^{(k)} > 0$$

and put $\delta = \text{Min}(\delta_1, \delta_2)$. For unlabelled nodes, change the node times (t_j) to ($t_j - \delta$). Discard all node labels other than (∞) and go back to step (2).

This iteration would ultimately terminate in step (2), when the terminal node (N) is labelled (∞) indicating that no more compression of project duration is possible. The direct project cost curve may be drawn from these calculations.

4.4 COMPUTATIONAL SHORT CUTS

Some original simplifications and modifications of Fulkerson's algorithm may be seen in an article by Roper [102], who has also borrowed ideas from Kelley. Roper's algorithm produces sub-project cost curves in addition the project cost curve. Several other extensions of the network flow approach of Fulkerson [39] have been published by different authors. The list includes Pareikh and Jewell [93]* who provide for decomposition of large networks into several subnetworks so as to reduce storage demands

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* Blanning and Rao [93] present a simpler method which does not require the use of Linear Programming techniques.

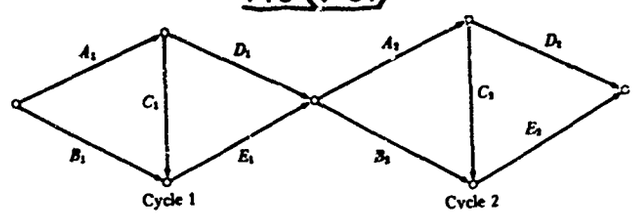
in machine computation, and Handa [50] and [51] who outlines computational short cuts by eliminating network segments which do not influence the project cost curves. In a different content, MacCrimmon & Ryavec [79] and Van Slyke [108] suggest similar computational short cuts (as outlined in Section 3 of this paper), while Fulkerson [41] & [42], discusses among other things, an iterative "longest chain" method for developing the project cost curve which can be used to delineate successive critical paths.

4.5 NODE BALANCING

If the convex, continuous, non-increasing activity cost curve were to be unbounded at both ends so as to resemble a hyperbola, as outlined by Clark [18], all activities in the optimum schedule become critical. Berman [4] relaxes the monotone assumption of Clark [18] by allowing the cost curve to turn upward above a certain activity duration. Using an 'event on node' type of network representation where several activities terminate at a given node while several others start from the same node, the nodes are balanced one by one by equating the sum of time-cost slope of incoming activities with that of the outgoing activities. The iterative algorithm terminates when all nodes are simultaneously in balance. He thus derives a minimum cost feasible schedule for each feasible schedule

for each feasible project duration which results in a continuous convex project cost curve. He applied this principle to a simple serial network with stochastic activity duration. He observes from this that, in general, uncertainity in activity duration tends to accelerate the occurrence of intermediate events, in particular, the ones which precede the uncertain activities.

FIG.(4.61)



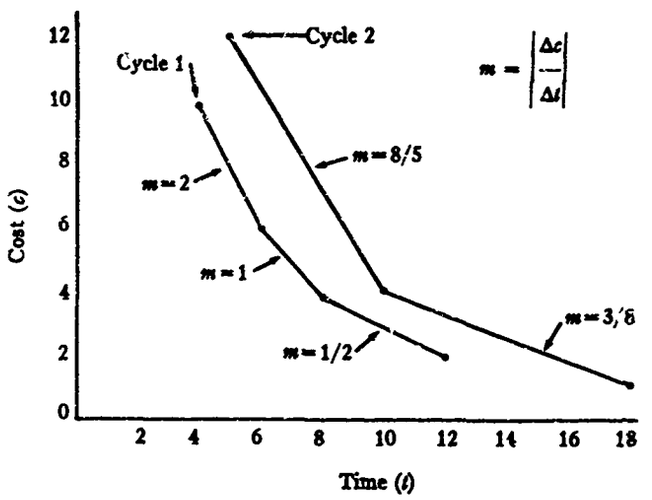
Two-Cycle Serial Network.

4.6 REPETITIVE PROJECTS

More recently, under the title of "Multicycle Project Planning", Fisher and Nemhauser [35] outline a method of arriving at a composite project cost curve Fig. (4.63) for a serial multicycle project Fig. (4.61) where the cost curve for each cycle Fig. (4.62) can be arrived at with the well established methods of Kelley [65] and Fulkerson [39]. In real situations, the repetitive operations are not strictly serial since portions of immediately succeeding cycles can begin when similar activities in the current cycle are completed. Such semi-parallel multi cycle projects lead to several complications in applying the

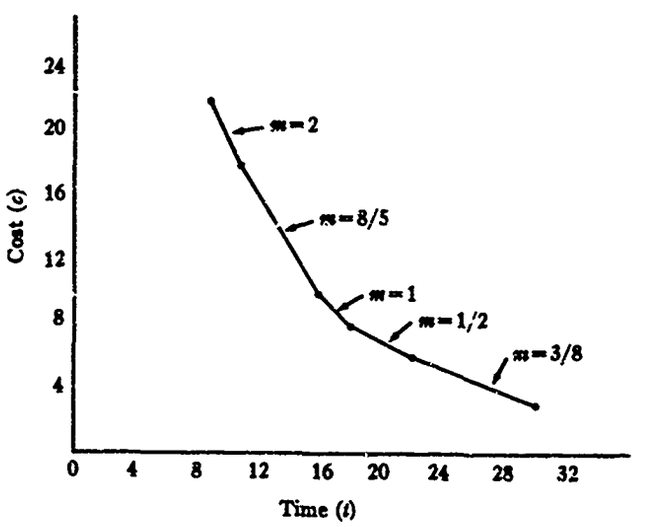
time-cost trade off principles, even though it is comparatively easy to identify the critical path. However, they provide an algorithm which generates the jobs and precedence relations of an entire multicycle project, if a corresponding description of a single cycle is given in an algebraic form (ie a set of integers representing events and a set of ordered pairs of integers from the event sets representing the activities).

FIG. (4.62)



Individual Cycle Project Cost Curves.

FIG (4.63)



Multicycle Project Cost Curve.

4.7 STRUCTURAL INTERPRETATION

While the few references cited above relate to the extensions of Fulkerson's Network Flow Algorithm, for the same restrictive assumptions of activity cost curve, an entirely different method of computing project cost curve is given by Prager [96]. The use of Fulkerson's approach necessitates the dual form of the orginal Linear Programming as given in Fig. (4.31), but Prager works within the framework of the original problem giving a structural interpretation to flow of cost in the network.

4.8 USE OF INTEGER PROGRAMMING

The restrictive assumptions of continuity and convexity inherent in Kelley, Fulkerson models are relaxed by several authors [1], [87], and most of them do not necessarily yield optimum solutions. However, Meyer and Shaffer [85] have come up with an integer-linear programming technique which can handle such cases for networks having fifty or less activities. As pointed out by Davis [24] and Carruthers & Battersby [14], the integer programming approach holds great promise for further developments. In the areas of network algebra, cost time trade offs and resource allocation, recent advances by Crowston and Thompson [22], Balas [2] and Pritsker and Watters [99] are based upon zero-one integer formulation. More references using the integer programming approach are cited in the

sixth section. Crowston has further extended his decision C.P.M. concept with several computational short cuts to include resource allocation models.

Decision C.P.M., (or D.C.P.M.) as given by Croston and Thompson [22] consists of a formal comparison of several ad hoc planning networks having different time, cost, resource or technological dependencies so as to choose those alternatives that minimize the total project cost. The authors recommend the repeated use of the same method during the execution phase of the project.

Using activity-on-node representation of a network and the notations given in Levy, Thompson and Wiest, Ch. (22), [66], they define a decision project graph G as a graph with nodes representing jobs and a directed time segment connecting two nodes S_{ij} (in job set S_i) and S_{mn}(in job set S_m), if and only if S_{ij} is an immediate predecessor of S_{mn} ie $S_{ij} << S_{mn}$. They define a decision project as a set J (a set of job sets $S_1, S_2, S_3, \cdots$) together with the specified interdependencies and the relation $<<$ defined on J. The outcome of decisions based on project cost analysis would be to eliminate a cluster of jobs from the decision project graph to obtain the final project graph used in a regular C.P.M. approach. This pruning may be achieved by several means such as complete enumeration, zero-one Integer Programming, Heuristics with

partial enumeration and branch and bound techniques. Crowston and Thompson illustrate with a few examples the several ways in which the 0-1 variable can be used to represent complicated interdependencies among decisions. The exponential growth of constraint sets as the number of activities are increased poses several practical problems some of which are currently being looked into by Crowston and others at M.I.T.

4.9 C.P.M. COST VS. C.P.M. TIME

While the papers cited above under "Cost-Time Trade Off" represent the mathematical developments in this area, the doctoral dissertation of Gessford [43] is devoted to a comparative study of CPM/cost and CPM/time. Using the data derived from questionnaires on housing construction as well as simulation, Gessford concludes that the utility of CPM/cost is correlative to project scope. Hence, medium and large size construction firms may find it economically and administratively advantageous to add cost constraints to their existing CPM/time systems. From the point of view of Accounting and Finance, CPM/cost promotes cash flow, budgeting and analysis, cost estimation and bidding, cost variance control and analysis, profit planning and the integration of project and fiscal planning.

4.10 RELIABILITY MATURITY INDEX

Finally, Malcolm, one of the four authors of the classic paper on PERT [80] extends the basic principles

of PERT into Reliability Management [81], coining a new term R.M.I. which represents Reliability Maturity Index. This descriptive paper outlines the procedure to be followed in measuring reliability in addition to time and cost data and including them in an integrated management system. The system makes use of information sampling of management reports to reduce the volume of input information.

SECTION FIVE: RESOURCE ALLOCATION: HEURISTIC MODELS

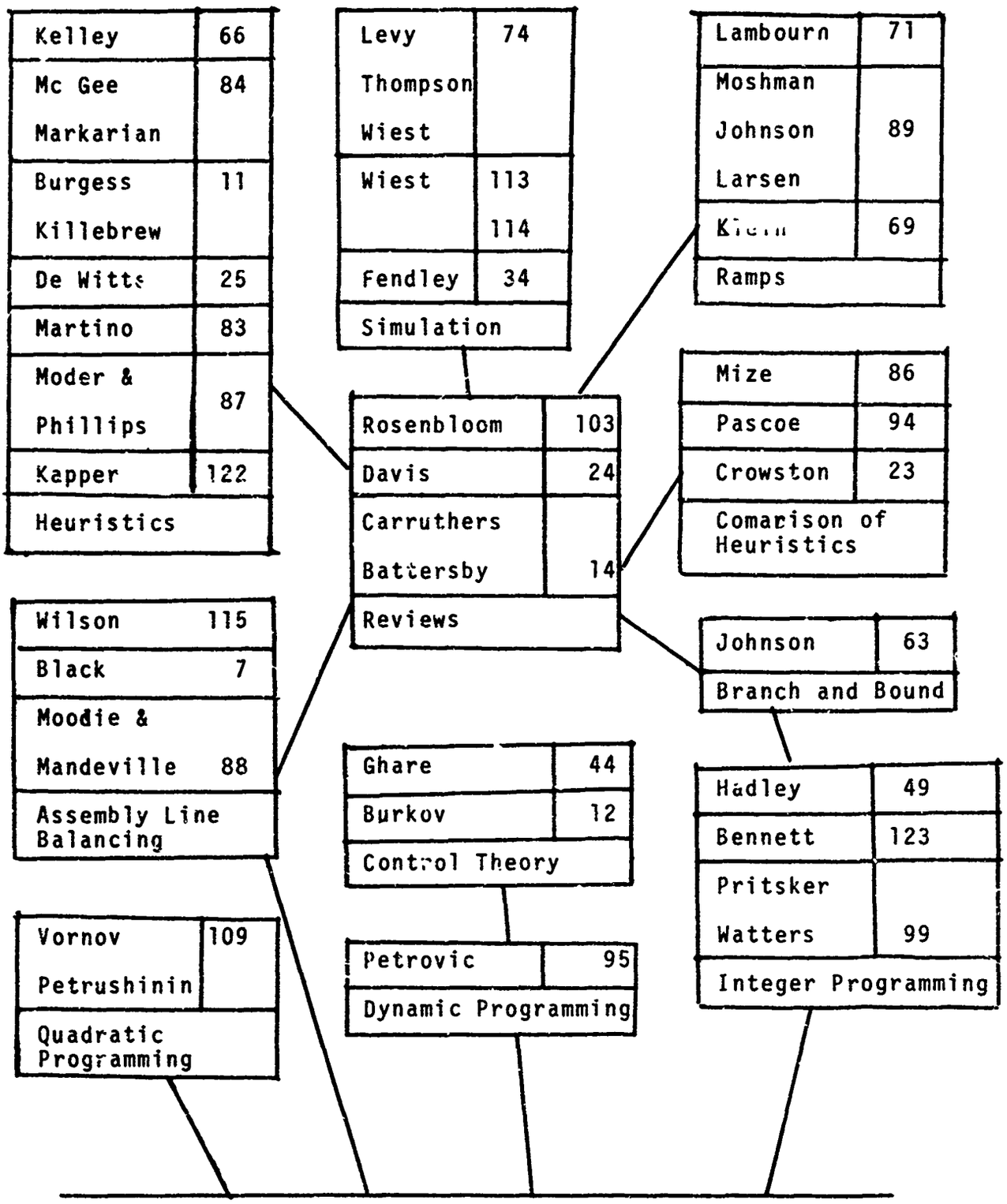
As in Section (4), the review articles of Rosenbloom [103], Davis [24] and Carruthers and Battersby [14] do good justice to the development of resource allocation models. Martino [83] has devoted an entire book to outline various heuristic models while Moder and Philips [87] present a few representative models. In all, these books and review articles cover most of the work done in this area and in particular, a comparative study by Pascoe [94] and by Mize [86] of different resource allocation models provides a sound basis for an objective analysis of this area which has currently become a focal point for further research. This is evidenced by recent published articles and a detailed review of these articles is given here.

5.1 THE COMBINATORIAL PROBLEM

The time-cost trade off principles outlined in the last section assume an unlimited availability of resources. Davis [24] recognizes the time-cost trade off procedures as one of the three special cases of resource allocation procedures, the other two being, the smoothing of resource usage if sufficient resources are available or the scheduling of activities when the resource availability is restricted to a predetermined maximum. Either way, as Kelley [66] points out, there are two major sources of difficulty--

Exhibit VIII

RESOURCE ALLOCATION



how should the problem of resource loading be formulated and how can it be solved? Irrespective of whether the resource required is labor, money, equipment or facilities, the existing practices in Project Management indicate a lack of explicit criteria for obtaining optimal use of resources. Hooper [55] regards this as a major problem area.

With regard to the mathematical formulation of the problem, difficulties arise because of its combinatorial structure. Kelley remarks:

> "At least, the several models we have considered are completely intractable. However, the difficulty seems to be intrinsic to the problem. The way the restrictions on the sequence in which jobs may be performed interacts with resource requirements and availabilities forces a solution set which is unconnected. This property exists even if we assume that men and equipment are infinitely divisible.
>
> Problems having unconnected solution sets are called combinatorial. Very few of them can be solved (in the mathematical sense) in a reasonable time. However, this does not imply that useful solutions cannot be obtained. Ad hoc intuitive methods, especially when used in conjunction with a computer to do the leg work,

can be quite effective for obtaining satisfactory solutions."

Carruthers and Battersby [14] summarize the state of art as follows:

"No algorithms for the resource smoothing problem exist which are rigorous, general and practicable; some exist which are rigorous, or rigorous and practicable but only in special cases. The rigorous approach to resource smoothing considers the entire project as one entity, and each decision in scheduling is made against the global requirements of the project. However, all practicable algorithms are intended to be reasonable rather than optimal, with regard to an objective function; consequently they are based on local decisions. Resource smoothing is currently a process of'meliorization' rather than 'optimization'."

5.2 CLASSIFICATION OF HEURISTICS

Mostly the heuristic models can be classified under series, parallel or series parallel methods of scheduling resources, as suggested by Kelley [66]. The original series method consists of ordering the list of activities by increasing slack within precedence constraints. By scheduling one activity at a time in the serial list, the precedence ordering guarantees that each task's logical

predecessors will have been scheduled previously. Thus, if sufficient resources are available, the activity in hand is scheduled to start at the earliest date, which follows the latest completion of the predecessor activity. In the parallel method, all activities that are ready to be scheduled are assembled on a given date. Using some intrinsic or explicit consideration as the case may be, each activity is examined and the best among them are scheduled if resources are available. The ones which are not scheduled are brought forward to the next period and examined along with the new activities which become available then. This operation is continued until all activities are scheduled.

While, in the series method, the original sequence is maintained throughout, there is no such identification of activity listing in the parallel method. The series parallel method begins by scheduling exactly as in serial method, but the sequence of activities remaining to be scheduled is reviewed at each step and altered if necessary, recognizing the changing circumstances which result from scheduling new activities.

5.3 A RESOURCE - TIME TRADE OFF MODEL

One of the early articles in this area is due to McGee and Markarian [84] who describe a serial method with variation in resource assignment and provision for

rescheduling. They consider a multi project situation where resources are allocated to different projects on the basis of the difference between the expected and desired project completion dates. They employ a procedure very similar to cost-time trade off principle outlined in Section (4). The desired schedule is determined by an iterative technique starting with minimal manpower allocation for each activity and successive manpower increments along the most productive critical activity. Even though the technique is attractive in view of its rather unique resource-time trade off structure, the lack of computer programs or computational results, the approach has not received due consideration by other researchers.

5.4 SIMULATION ALGORITHMS

On the other hand, Levy, Thompson and Wiest[74] also consider a multi project situation in a job shop with an elaborate computer program which minimizes the peak load over a future time interval, treating resource upper limit as a variable rather than as a restraint. A novel feature of this method is the use of random choice steps in rescheduling slack activities, thereby necessitating repeated trials inherent in simulation techniques, so as to arrive at a satisfactory schedule. As Davis [24] points out, this does not always produce optimal results,

since its effectiveness will vary with the number of non critical activities and their slack values.

One of the authors of the article mentioned above, J. D. Wiest [114], takes another look at the original problem as one allocated on the basis of a group of scheduling rules rather than as a resource smoothing problem. In the basic model, in any given period, the currently available activities are listed according to their total slackness such that the most critical activities have the highest probability of being scheduled first. Using Monte Carlo, as before, the activities are scheduled up to the resource limit. The unscheduled jobs are pushed forward and when they eventually become critical, they receive top priority in the current activity list.

As of today, Wiest's model is the most promising one in view of its flexibility to include a variety of realistic system attributes. For instance, his sub-routines permit splitting and staggering of activities, trade offs between time and resource for individual activities, use of multiple resources, transfer of resources between activities and pre-emption of activities as well. The model, thus, proceeds with scheduling and rescheduling each day optimizing locally a complicated cost function based on resource utilization and expected duration. His cost function

includes costs of unused resources and premium and overtime resources which are affected by the schedule, overhead costs related to project duration, hiring and lay off costs which change the resource levels and so on. His model is, in part, based on the concepts of "critical sequence' and "conditional slack' derived from a limited-resource modification of network scheduling methodology, as given in [113].

5.5 OTHER HEURISTIC MODELS

Being the most recent one, Wiest's model represents a substantial improvement over several heuristic models in resource allocation. Among them are the contributions by Burgess and Killebrew [11] who use the sum of squares of resource requirements as a measure of effectiveness while DeWitte [25] minimizes the absolute magnitude of flunctuations from a calculated project mean level of resource usage.

Other approaches at multi project level include the work of Mize [86] who developed a new heuristic algorithm for scheduling activities to several operating facilities. Martino [83] has developed a heuristic technique called M.A.P. to consider multi project resources with fixed or variable or a combination of fixed and variable limits. His technique prescribes no upper limit either to the number of resource types or to the number of activities.

Yet another system available in the market is due to C.I.E.R., a computer service organization, who offer a comprehensive package caled RAMPS [71]. Being a proprietary system, details of the technique are not available. The system is believed to be heuristic and performs point-wise optimization by ranking each of the feasible subsets of currently available activities on the basis of a complex function which includes work continuity, project delay, idle resource, etc. Another recent article on RAMPS II by Klein [69] provides some more insight into the critical Path Algorithm, describing error detection and network analysis features. It can operate rapidly, since no topological ordering of the network needs to be performed. It can process randomly numbered or even alphanumerically named nodes.

As in RAMPS, several computer programs involving the total systems approach as well as special purpose programs are reported at the 29th annual meeting,ORSA, Houston, Texas (Nov. 1965). Among them are systems such as U.P.S.S., due to Metzger and Thiele and SCOPE, due to Miller and Miller. In 1964, Moder and Philips [87] reported the details of 36 out of 60 or more programs then available. A similar listing of currently available programs would indeed be very lengthy and perhaps less useful.

5.6 COMPARATIVE STUDIES

Many researchers in this field have been keenly interested in a comparative study of different procedures. For instance. Pascoe [94] has examined both series and parallel algorithms with a wide range of characteristics involving project logic, look ahead features, time and or resource constraints, network complexity and size, multiplicity of resources and with five different objective functions. His conclusions are that the parallel methods of resource smoothing are slightly (ie at 0.1% probability level of significance) superior to the serial method and that the use of the latest start for large networks and the latest finish for small networks are better than early start or early finish routines. Otherwise there is hardly any difference between the heuristics employed.

Crowston [23] has developed several ranking criteria including the "greatest remaining resource utilization", "remaining slack", etc. and suggests several ad hoc rules for pre-empting activities. The results of his study are still inconclusive.

Mize [86] has also made a comparative study of selected heuristic algorithms. The number of test cases constructed by him were too small to demonstrate convincingly that any one method was superior to another.

In testing and evaluating heuristic models, Wiest [114] suggests three feasible approaches: laboratory tests as outlined by Pascoe, field tests where a given model is applied to a variety of real world scheduling problems under differeent scheduling environments and constraints and finally comparative tests where several computer models and practical methods are compared for the same real projects. Much work needs to be done in each case before objective conclusions can be reached.

5.7 RESOURCE SCHEDULING FOR STOCHASTIC ACTIVITIES

The results reported so far refer to single or multiple projects having deterministic activity durations. The doctoral dissertation of Fendley [34] on the other hand concentrates on the multi project resource scheduling given that the activities have stochastic duration of a PERT type. Using the Monte Carlo Technique on a combination of hypthetical projects each having up to 20 activities, three resource types and 200 iterations and eight different priority rules, Fendley simulates a project scheduling problem. The two most significant conclusions from the experiment are:

1. The minimum-slack-first (M.S.F.) priority rule, when used with realistic due dates is the best of those rules tested.

2. Realistic due dates may be set by an analysis of resource load on the facility to determine the amount

of slippage that must occur to perform all projects with fixed resources.

Under the M.S.F. rule, an activity is loaded from a queue with a priority dependent upon both project characteristics and the relative urgency of the project's due date. Using regression analysis on the experimental data, Fendly predicts slippage necessary for the performance of all projects in the hypothetical system, so that the system can be loaded to any desired capacity. Even though it is hard to generalize from this limited experimental investigation, the approach of Fendley is rather unique and promising.

5.8 PRACTICAL STUDIES IN COMPUTATIONAL FEASIBILITY

Another recent doctoral dissertation by Kapper [122], who applied the Critical Path Analysis to health and hospital fields, provides methods for handling several practical and computational difficulties. Among them are: (a) the rapid and practical segmentation of any plan into manageable timesteps (b) the comparison of the resource scheduled for use with those actually available at any point of time (c) the generalization of a schedule where the resources which are used approach but not exceed the resource limits specified for each point of time (d) the rapid practical integration of this algorithm and its associated computer programs with other currently available

scheduling programs (e) a variety of distribution functions for characterizing activity completion times (f) the bias associated with time/cost/resource requirement estimates (g) the production of an on-line real time, planning-scheduling-allocation capability consistent with the contingency and feed back requirements of plan implementation; and (h) the incorporation of auxiliary capabilities (e.g. decision table logic, trend analysis and information retrieval) in order to achieve sub optimal allocation (Thesis Abstract).

SECTION SIX: RESOURCE ALLOCATION

The papers discussed so far in the area of resource allocation have been mostly heuristic, with particular reference to elaborate schemes outlined by Wiest [114], Fendley [34] and Kapper [122]. However the most recent work in this area is directed towards a mathematical formulation of the problem. In brief, the review includes a significant contribution by Johnson [63] who applies the principles of branch and bound for simple models, the work of Pritsker and Watters [99] who suggest a zero-one programming approach, a resource-time trade off model of Ghare [44], an excellent series of Russian and Yugoslav papers [109], [12], [95] covering the areas of quadratic programming, dynamic programming and the optimal control theory. This section also contains the dissertation abstract of Bennett [123] who reports an Integer Programming approach and a series of contributions relating the assembly line balancing to the resource allocation problem.

6.1 BRANCH AND BOUND IN RESOURCE ALLOCATION

The branch and bound idea was used by with Little, et al [75] while they were seeking a manageable solution for the travelling salesmen problem. Lawler and Wood [73] provide a survey of several papers in this area, illustrating the wide applicability of this concept in several O.R.

problems. Johnson's thesis [63] is yet another illustration of this simple but powerful method of branch and bound. Using both critical path and resource requirement data he outlines a minimum bounding technique which automatically discards several partial schedules which cannot possibly improve upon a known complete schedule. He defines such an elimination procedure in terms of a "limited enumeration" algorithm. He also provides for a comparison among partial schedules themselves thereby reducing substantially more computational work

These "partial schedule dominance" tests are the added features of his computer program which is workable up to 50 jobs. Beyond this range, the computational time required become too large to be economical. However, the branch and bound approach may still provide a meaningful approximations to large complex problems. Depending upon the nature of the problem, the use of appropriate bounding and dominance tests along with an efficient data organization should considerably improve upon the present state of the art.

6.2 A ZERO - ONE INTEGER PROGRAMMING FORMULATION

Pritsker & Watters [99] present amathematical approach to scheduling problems in general with the help of a zero-one integer programming formulation. They demonstrate that it can accomodate a wide range of real world situations

and perhaps large practical problems may eventually be solved in this manner. In particular, the 0-1 variables are used to indicate for select periods whether or not a job has been completed. The period selection depends upon the job arrival time, due dates, sequencing relationships, etc. They consider three objectives: minimize total throughput time for all projects; minimize the time by which all projects are completed and finally minimize total lateness or lateness penalty for all projects. The constrained set includes limited resources, precedence relations between jobs, job splitting possibilities, project and job due dates, resource substitution, concurrent and non concurrent job performance requirements. With the help of simple theoretical examples, they demonstrate the superiority of 0-1 formulation over known heuristics. This attractive theoretical formulation remains to be tested for large practical problems with regard to its computational feasibility even as Johnson's "branch and bound" [63], Meyer and Shaffer's Cost-Time Trade Off [85] and Crowston and Thompson's Decision C.P.M. [22] have all encountered similar difficulties. Yet another theoretical approach to resource allocation problem using integer-programming may be seen in Hadley [49].

6.3 ANOTHER RESOURCE - TIME TRADE OFF MODEL

As suggested by Clark [18], Berman [4], Jewell [60]

and McGee and Markarian [84], the work of Ghare [44] is based on the assumption that the duration of an activity can be controlled by the modification of resources allocated to the activity. Time and Resource are considered the through variable and the across variable of the allocation system respectively. Using the techniques of the systems theory, and a mathematical programming algorithm, the project duration is minimized. The major emphasis is on the management control of activity duration, while proceeding with an optimal allocation of scarce resources.

6.4 A QUADRATIC PROGRAMMING APPROACH

Significant among the analytical contributions on Resource Allocation are the works of Voronov and Petrushinin [109] who use Beale's Quadratic Programming method in minimizing simultaneously the mean square deviation of the resource allocation function from a constant. The allocation of resources $Y_{ij}(t)$ to activities q_{ij} may become non-uniform, changing by jumps at certain time intervals while remaining constant in between, satisfying the condition

$$Y_{ij}(t_\alpha + 0) = \begin{cases} \equiv 0 \ , & \text{for} \quad \alpha < i \\ \geqslant 0 \ , & \text{for} \quad i \leq \alpha \leq j \\ \equiv 0 \ , & \text{for} \quad \alpha > j \end{cases}$$

where t_α is the time of occurrence of event α.

The model includes the possibility that the allocation

function $Y_{ij}(t)$ reduces to zero during any of the intermediate intervals. An important assumption of the model is that the moments at which events occur are assumed to be fixed and consequently, the interval between event times ($t_{\alpha+1} - t_{\alpha}$) are also fixed. Therefore, for every activity, there is a linear algebraic equation

$$\sum_{\alpha=0}^{n-1} (t_{\alpha+1} - t_{\alpha})\, Y_{ij}(t_{\alpha}+0) = Q_{ij} \quad \text{for}$$

$(i<j)$ and ($i,j = 1,2,\ldots n$), where Q_{ij} is the initially given work content of the activity. The objective function has a quadratic form

$$\Pi = \sum_{\alpha=0}^{n-1} (t_{\alpha+1} - t_{\alpha}) \left[\sum_{q_{ij} \in P_s} Y_{ij}(t_{\alpha}+0) \right]^2$$

where the activity q_{ij} is a member of the subset P_s of the work performed by resources of the sth type. With the help of a simple theoretical example, Voronov and Petrushinin show all the steps involved in using Beale's Method of Quadratic Programming.

6.5 USE OF OPTIMAL CONTROL THEORY

Burkov [12] considers a class of resource distribution problems from the standpoint of the optimal control theory. He regards the process of accomplishing a project as a transient process in the automatic control

system where the resources play the role of control actions. The dimension of phase space of the system state is equal to the minimal number of independent paths of the project network (ie the paths having no common nodes). Burkov confines his attention to the problem of cost minimization within a restricted project completion time. He shows that the convergence problem in the iteration process involved in searching for an optimal solution is reduced to analysis of system stability in a transient process of automatic control system.

6.6 A DYNAMIC PROGRAMMING FORMULATION

Burkov's control-theoretic approach has been limited to cost minimization. Pursuing his ideas further, Petrovic [95] maps the predecessor-successor relations of activities $\{(i,j) \in A\}$ into a set of transforms

$$X_{ij}(n+1) = T_{ij}\left\{X_{ij}(n),\ U_{ij}(n)\right\}, \quad (i,j) \in A, \quad n = 0, 1, \ldots N-1$$

where the state variable is the scope of work $X_{ij}(t)$ required to complete an activity and the decision variable is the resource $U_{ij}(t)$ allocated to each activity. If the availability of a resource is limited to U, then $\sum_{(i,j)\in A} U_{ij}(n) \in U$, $n = 0, 1, \ldots N-1$. The admissible set U may be regarded as a given function of time or the state of the project itself. Further using the network constraints

$$\left.\begin{array}{ll} U_{jk} = 0 & \text{for } x_{ij} \neq 0 \\ U_{jk} = 0 & \text{for } x_{jk} = 0 \end{array}\right\} \text{ for } \begin{array}{l} (i,j) \in A \\ (j,k) \in A \end{array} \text{ and } i < j < k$$

and scalar objective function,

$$J = \sum_{(i,j)\in A} \sum_{n=0}^{N} g_{ij}\left(X_{ij}(n),\ U_{ij}(n)\right)$$

Petrovic views this multistage decision process as a dynamic programming problem having the following recurrence relationship in the case of a single resource.

$$f_{N-K}\left(X_{ij}(N-K,\ N-K+1)\right) = \min_{U_{ij}\in\left[(N-K),(N-K+1)\right]} \left\{ \left[U_{ij}(N-K)\right]^2 + f_{N-K+1}\left[T\left(X_{ij}(N-K)\right)\right]\right\}$$

for $K = 2, 3, \ldots N$

with

$$f_{N-1}\left\{X_{ij}(N-1,N)\right\} = \min_{U_{ij}\in(N-1,N)} \left[U_{ij}(N-1)\right]^2$$

Inevitably, the dimensions of the functional equations which appear in the model become very high in practice, thereby presenting serious limitations in computational feasibility. In reducing the necessary amount of computational effort, Petrovic suggests three sophisticated approaches. The first approach is called sub-project programming which involves an ingenious time-wise decomposition of a time-scale network. Selecting sub-projects in this manner, every crossing state variable between two adjacent sub-projects is considered as an output of one

of them and an input of the other. An optimum aggregation of all sub-projects can then be made with the help of the crossing variables in such a way that the aggregate solution is equivalent to that of the original network.

The second approach is again another form of decomposition employing a 'two level concept' which is based on the results of Lasdon and Schoeffler [72]. The two level scheme consists of the decomposed component subsystems to each of which is assigned a specially constructed, independent sub-problem. The objective function of each sub-problem contains a number of variable parameters or "prices" whose values are determined by a second level controller. Sub-problem solutions are functions of these parameters. Lasdon and Schoeffler give the criteria which enable the second level to find the "optimal" parameter set by a simple iterative procedure.

The last approach represents an approximation in policy space as given in Bellman and Dreyfus [3]. The procedure consists of a simple initial approximation to the optimal policy. Then a control policy satisfying certain properties is obtained such that it yields a better performance. The control policy is then changed partially to effect optimal improvement at each iteration. When the value of the criterian decrement becomes less than the specified tolerance, the iterations may stop,

even though computations may be stopped much earlier if desired.

Petrovic's paper represents a significant departure from the familiar heuristic approach to the resource allocation problem. However the ideas presented here are yet to be validated by specific illustrations which would demonstrate their computational feasibility.

6.7 COMPUTATIONAL EXPERIENCE WITH DIFFERENT APPROACHES

Yet another integer programming approach to the resource allocation problem is given in an unpublished doctoral thesis of Bennett from Cornell University [123]. An excerpt from his thesis abstract is given below.

> "This thesis develops several approaches to the following problem: Sechdule the activities of a project network, some or all of which require resources which are available in limited amounts, in such a way that the activities are performed in proper sequence, the resource availabilities are not exceeded, and the duration of the project is as short as possible.
>
> Mathematical models are presented which, when solved, can assure that these conditions are met. A linear model represents the case in which activity durations are considered to be fixed, and the resources of constant amounts are assigned to each

activity. A linear model, a quadratic model, and a transportation format ranking procedure are given for the situation in which activity durations may vary for different resource assignments and only total resource requirements are given for each activity.

The two linear models require integer linear programming for solution. Experience in attempting to solve the formulations with two different computer codes on a Control Data 1604 computer is reported. Although this experience was valuable in terms of testing a rather new mathematical technique, it is clear that this approach could not be applied to the planning of actual projects because of limited computer memory capacity and excessive run time. For reasons that are not clearly understood, all attempts to solve the quadratic model by a method suggested by Wolfe were unsuccessful. The final technique, a ranking procedure, is limited to only narrow applications.

Because of the limitations of the minimum time models, an heuristic algorithm is given for the case in which activity durations may vary. This program essentially works through a list of activities, assigning resources to them for each time unit of the

project until all resources available during that time units are assigned or until no more activities can be scheduled. Consideration then turns to the next time unit, and the procedure is repeated. When all activities have been assigned their total resource requirements, the project is considered complete, and the program terminates. The resulting schedule is feasible, since resource availabilities are not exceeded, but not necessarily optimal, since the duration may not be as short as possible.

The program can schedule projects containing as many as 700 activities and having durations as great as 400 days. Seven resources may be scheduled simultaneously. In addition to the total requirement for each resource, the input for each activity includes the maximum amounts of each resource that can be assigned during any time period and the minimum amount that can be assigned if the amount is greater than zero. Experience with the program indicates that it can be applied successfully in scheduling reasonably large projects."

6.8 ASSEMBLY LINE BALANCING PROBLEMS

Brief descriptions of resource allocation models of Wilson [115] and Black [7] may be seen in the review article by Davis [24]. Apart from suggesting a dynamic programming approach, Wilson, for the first time, provides a linkage between resource allocation problems and assembly line balancing problems as outlined by Hu [59]. Proceeding on similar lines, Black [7] adapts Gutjahr-Nemhauser algorithm for assembly line balancing [48]. More recently, the similarities between these two techniques are established by Moodie and Mandeville [88]. Their main source is Bowman's [9] presentation of an Integer Programming method of assembly line balancing. Moodie and Mandeville indicate the basic differences between the two problems and illustrate their approach by obtaining a computer solution to a simple theoretical problem. The merits of heuristics for multiproject resource balancing are also discussed.

SECTION SEVEN: DIRECTIONS OF RESEARCH

In one of his early papers (1961) describing the primal-dual algorithm for the time-cost trade off problem, Kelley [65], p. 314, remarked:

> "Quite often large portions of a project diagram form individual projects in their own right. They connect into the whole project at only two points, their origin and terminus, respectively. Each of these sub-projects can be solved separately, a spectrum of solutions and utility functions being obtained in each case. Each sub-project may now be replaced in the project by a single activity with the same utility function as the sub-project it replaces. The original project is now smaller by many activities and events and can be solved more easily than before."

The work of Parikh and Jewell [93] (1965) on decomposition of project networks essentially achieves this goal, using Fulkerson s [39] algorithm. While providing a solution to a project consisting of sub-projects which are connected in series or parallel or a combination of both, they recognize that the "generalized clustered network" is more difficult and no simple algorithm is

available. They suggest the use of "out-of-kilter" method [37] for each interlink and each desired project duration. In a different context, Jewell [61] has made use of network flow approach to include "divisibility" of activities, suggesting the need for techniques in the area of multi-commodity network flows if two or more activities are divisible. He has also hinted the possibilities of extending this model to time-cost trade off area. More recently, Jewell [62] has made further contributions to the "out of kilter" algorithm while progress is being made in the area of multi-commodity network flow [125] and project decomposition [78] extending the computer program to PERT type of networks. Yet another stream of thoughts pertaining to stochastic activity duration relates to the chance constrained program [16] and directed sub-dual algorithm [15] for time calculations only, leaving room for development in the time-cost trade off area. The synthesis of these ideas may also include the feedback stochastic networks of Elmaghraby [32], [33], Pritsker and others [97], [98], [112], [117]. Finally the developments in "Decision C.P.M." due to Crowston and Thompson [22] could be merged with others providing for extensions in decomposition of integer programs [124], branch and bound applications [63], the use of filter method [2] and disjunctive constrains [2] and [104].

Klingel [70] observed that while resource allocation programs reduce the deterministic completion time by increasing the parallelism of the network, the probability of project completion by the new date is also reduced. On the other hand, Wiest [113] shows that the usual concepts of 'critical path' and 'job slack' lose their meaning in the scheduling for large projects with limited resources. He suggest the use of a concept called 'The Critical Sequence' which identifies a sequence of critical jobs composed of either a technological sequence, or a resource sequence or both. In a similar manner, Van Slyke [108] introduces the concept of "criticality index' while Noettl and Brumbaugh [91] introduce information theoretic concepts in network planning. In the light of these new concepts, and the mathematical developments using control theory [12], quadratric programming [109] and integer programming [99], [49], [123] for resource allocation problems, further strides can be made to integrate time-cost trade off with resource limitations and perhaps a new dimension involving performance standards. Equally challenging are the fast developing areas of assembly line balancing and production scheduling where the similarities with the project networks have already been established: [115], [7], [88] and [107].

From a practical point of view, the contributions of King and others [67] and [68] need to be substantiated by further data about the estimating behavior of individuals while a meaningful synthesis of Line of Balance and PERT networks (PERT/LOB) as suggested by Schoderbek and Digman [105] extend these concepts to repetitive jobs.

In conclusion, the exhibit IV attempts to provide a basis for cross fertilization among the four areas of network algebra, PERT assumptions, cost-time trade off and resource allocation. The applicability of different mathematical programming and other techniques in this area indicates that developments in these respective areas may have an influence upon the project networks and their computer programs.

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13. ABSTRACT This paper brings together different aspects of the Critical Path Analysis exclusively in terms of the mathematical developments in this area during the last eight years. The survey is divided into seven sections. The first one deals with the four stages of development, each successive stage being more representative of the real situation. The next five sections give explicit reference to the different articles which contribute to the stages of development described in the first section. The articles are reviewed in terms of the network algebra, PERT assumptions, cost/time trade off, resource allocation (heuristic) and resource allocation (analytical). Finally, the seventh and last section of this paper provides a summary of ideas and possible directions of research, some of which are currently being undertaken.

14. KEY WORDS

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