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## Technical Report

317

### Membrane and Bending Stresses in Shallow, Spherical Shells

F. Y. M. Wan

11 August 1964

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## Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts



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MEMBRANE AND BENDING STRESSES  
IN SHALLOW, SPHERICAL SHELLS

*F. Y. M. WAN*

*Group 71*

TECHNICAL REPORT 317

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ABSTRACT

The present work is concerned with the nature of the interior solution and the influence coefficients of shallow, spherical, thin, elastic shells (or equivalently a shallow, thin, elastic, paraboloidal shell of revolution) which are homogeneous, isotropic, closed at the apex, and of uniform thickness. The investigation is carried out within the framework of the usual shallow shell theory for small displacements and negligible transverse-shear deformations. Exact interior solutions are obtained for shells acted upon by edge loads and edge moments. The constants of integration associated with these interior solutions are expanded asymptotically in inverse powers of a large parameter. Retaining only the leading term of these expansions leads (in most cases) to known approximate results. Explicit expressions for the second-order terms are obtained. It is shown that these second-order terms play a significant role in a certain class of problems. The relative importance of the membrane and inextensional bending stresses in the interior of the shell is discussed. The exact and asymptotic influence coefficients are obtained. The interior stress state of shells subjected to polar harmonic axial surface loads is also investigated by the same procedure.

Accepted for the Air Force  
Franklin C. Hudson, Deputy Chief  
Air Force Lincoln Laboratory Office

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\*The major portion of this report is based on a thesis submitted to the Department of Mathematics at the Massachusetts Institute of Technology on 17 May 1963, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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## NOMENCLATURE

$x, y$	Cartesian coordinates in base plane (plane tangent to apex of undeformed middle surface of shell) with origin at apex
$z$	Distance between base plane and a point on undeformed middle surface
$r$	Distance between a point on undeformed middle surface to axis of revolution
$\theta$	Polar angle in base plane measured from positive x-axis
$\vec{i}_\theta$	Circumferential base vector of undeformed middle surface
$\vec{i}_r$	Meridional base vector of undeformed middle surface
$\vec{n}$	Unit normal of undeformed middle surface, positive inward
$F$	Stress function from which in-plane stress resultants are derivable
$w$	Transverse middle surface displacement
$u$	Meridional middle surface displacement
$v$	Circumferential middle surface displacement
$N_r, N_\theta, N_{r\theta}$	In-plane stress resultants
$M_r, M_\theta, M_{r\theta}$	Moment resultants
$Q_r, Q_\theta$	Transverse-shear resultants
$\beta_r, \beta_\theta$	Rotations of middle surface
$P_n$	Normal component of surface load intensity
$P_r$	Meridional component of surface load intensity
$P_\theta$	Circumferential component of surface load intensity
$P_z$	Component of surface load intensity in direction of axis of revolution, i.e., the axial component
$D$	Bending stiffness factor
$1/A$	Stretching stiffness factor
$\nu$	Poisson's ratio
$h$	Wall thickness of shell
$E$	Young's modulus
$a$	Value of $r$ at edge of shell
$(\ )_{,r}$	$= \frac{\partial(\ )}{\partial r}$
$(\ )_{,\theta}$	$= \frac{\partial(\ )}{\partial \theta}$
$\nabla^2(\ )$	$= (\ )_{,rr} + 1/r(\ )_{,r} + 1/r^2(\ )_{,\theta\theta}$

$\sigma_B$	Bending stress
$\sigma_D$	Direct or membrane stress
$( )_p$	Particular solution for quantity in bracket
R	Radius of spherical middle surface or twice focal length of paraboloidal surface of revolution

# MEMBRANE AND BENDING STRESSES IN SHALLOW, SPHERICAL SHELLS

## I. INTRODUCTION

One of the characteristic behaviors of shell structures is the coupling between their bending and stretching actions. Through this coupling, a shell offers more resistance to transverse loadings than, say, a flat plate of the same dimensions and with the same material properties. However, accrued to this structural advantage is the price of mathematical complexity. Even in a linear theory, it is often necessary to introduce additional assumptions based on plausible arguments, if the statics problem of thin elastic shells is to lend itself to manageable solutions.

The membrane theory of shells, for instance, assumes that, away from its edge(s), a shell prefers to carry the applied loads by the development of the in-plane stress resultants rather than by the transverse-shear and moment resultants; hence, an approximate interior solution can be obtained by neglecting terms associated with the bending action in the differential equations of equilibrium. Such a procedure does, in fact, lead to a good approximation of the interior stresses for several types of shells of revolution under rotationally symmetric loads, and loads varying sinusoidally in the circumferential direction with period  $2\pi$  (Refs. 1-3). On the other hand, the correspondence between the interior stress state and the applied loads recently established by E. Reissner<sup>4,5</sup> for shallow, spherical shells shows that this is not always true for self-equilibrating loads. More recently, the same correspondence for shells of revolution was discussed by C. R. Steele.<sup>6</sup>

The present work pursues further this subject of membrane vs inextensional bending interior stress state and examines a larger class of boundary value problems associated with small deformations of shallow, spherical, thin, elastic shells (which is the same as shallow, thin, elastic, paraboloidal shells of revolution) than those explicitly investigated hitherto. My analysis differs from the earlier writers in that the boundary value problems will now be solved exactly to obtain the constants of integration associated with the interior solution in terms of the geometrical and material properties of the shell and in terms of the prescribed loads and/or constraints. These constants will then be expanded in inverse powers of a large parameter to unveil the nature of the interior stress state.

Some known results for small deformations of shallow, spherical shells<sup>5,7-9</sup> needed in the subsequent development are recapitulated in Sec. II. Sections III, IV, V and VI deal with shells without surface loads. The shells are subjected to edge loads and edge moments in such a way that the over-all static equilibrium of the shell is maintained. In Sec. III, shells with prescribed edge loads and edge moments are considered. Once the asymptotic interior solution is established, a comparison of the corresponding membrane and inextensional bending stresses shows how the nature of the interior stresses may vary with the prescribed quantities. The

exact and asymptotic influence coefficients are then obtained. To do this, we need the explicit expressions for the constants of integration associated with the edge zone solution. Since we have the explicit expressions for all the constants of integration, the stress boundary value problem is now completely (and exactly) solved. As a side result, it is shown that the membrane and inextensional bending solutions, obtained by way of a set of contracted stress boundary conditions established by M. W. Johnson and E. Reissner<sup>10</sup> for shallow, spherical shells, and later by E. Reissner<sup>11</sup> for general shells, are the leading term of the asymptotic interior solution. Shells with prescribed edge deformations are studied in Sec. IV. Section V deals with shells whose tangential edge displacements and transverse edge load and moment are prescribed. Shells with a different type of mixed boundary conditions are treated in Sec. VI. There, the tangential edge loads and transverse edge deflection and moment of the shells are prescribed. We then turn to shells with surface loads. In Sec. VII, shells subjected to axial surface loads in the form of polar harmonics are considered. The nature of the interior stress state for shells with various types of edge support are investigated.

## II. FORMULATION OF PROBLEM

### A. Differential Equations in Polar Coordinates

The system of differential equations governing the small deformations of an isotropic, shallow, spherical shell with constant wall thickness and negligible transverse-shear deformability is<sup>7-9</sup>

$$\begin{aligned} D\nabla^2\nabla^2w - \frac{1}{R}\nabla^2F &= P_n + \frac{2\Omega}{R} \\ A\nabla^2\nabla^2F + \frac{1}{R}\nabla^2w &= -A(1-\nu)\nabla^2\Omega \end{aligned} \quad , \quad (\text{II-1})$$

where

- w = the transverse component of the middle surface displacement,
- F = a stress function representing the direct stress resultants,
- R = the radius of the spherical middle surface,
- P<sub>n</sub> = the transverse component of the surface load intensity vector,

$$\nabla^2(\ ) = \frac{\partial^2(\ )}{\partial r^2} + \frac{1}{r}\frac{\partial(\ )}{\partial r} + \frac{1}{r^2}\frac{\partial^2(\ )}{\partial \theta^2} = (\ )_{,rr} + \frac{1}{r}(\ )_{,r} + \frac{1}{r^2}(\ )_{,\theta\theta} \quad ,$$

1/A = E<sub>s</sub>h, the stretching stiffness factor,

D = E<sub>b</sub>h<sup>3</sup>/12(1-ν<sup>2</sup>), the bending stiffness factor,

ν = Poisson's ratio,

h = shell thickness,

r, θ = polar coordinates in the plane tangent to the apex of the shell,

and where it is assumed that the meridional and circumferential components P<sub>r</sub> and P<sub>θ</sub> of the surface load intensity vector are derivable from a load potential Ω in the form

$$P_r = -\frac{\partial\Omega}{\partial r} \quad , \quad P_\theta = -\frac{1}{r}\frac{\partial\Omega}{\partial\theta} \quad . \quad (\text{II-2})$$

Some degree of nonhomogeneity is included in the above formulation by allowing an independent choice of bending and stretching stiffness factors. If the shell is completely homogeneous, then

$E_b = E_s = E$ , where  $E$  is Young's modulus. The geometrical properties of the shell are shown in Fig. 1.

The relevant stress resultants and couples are given in terms of  $F$  and  $w$  by the following relations (Fig. 2):

$$\begin{aligned}
N_r &= \frac{1}{r} F_{,r} + \frac{1}{r^2} F_{,\theta\theta} + \Omega \quad , \\
N_\theta &= F_{,rr} + \Omega \quad , \quad N_{r\theta} = -\left(\frac{1}{r} F_{,\theta}\right)_{,r} \quad , \\
Q_r &= -D(\nabla^2 w)_{,r} \quad , \quad Q_\theta = -\frac{D}{r} (\nabla^2 w)_{,\theta} \quad , \\
M_r &= -D \left[ w_{,rr} + \nu \left( \frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta} \right) \right] \quad , \\
M_\theta &= -D \left[ \nu w_{,rr} + \left( \frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta} \right) \right] \quad , \\
M_{r\theta} &= -D(1-\nu) \left( \frac{1}{r} w_{,\theta} \right)_{,r} \quad .
\end{aligned} \tag{II-3}$$

The meridional and circumferential middle surface displacement components  $u$  and  $v$  are related to  $F$  and  $w$  as follows:

$$\begin{aligned}
u_{,r} - \frac{w}{R} &= A(N_r - \nu N_\theta) \quad , \\
\frac{1}{r} v_{,\theta} + \frac{1}{r} u - \frac{w}{R} &= A(N_\theta - \nu N_r) \quad , \\
\frac{1}{r} u_{,\theta} + r \left( \frac{1}{r} v \right)_{,r} &= 2(1+\nu) AN_{r\theta} \quad .
\end{aligned} \tag{II-4}$$

## B. Boundary Conditions

To complete the description of the problem, we must supplement the differential equations with an appropriate set of boundary conditions. Throughout the present investigation, the shell is to be closed at the apex and is to extend over the region  $0 \leq r \leq a$ ,  $0 \leq \theta \leq 2\pi$ . At the apex  $r = 0$ , the shell is to have finite stresses and displacements. At the edge  $r = a$ , four consistent and independent conditions must be prescribed. Among the possible combinations of edge conditions are the stress conditions

$$N_r = \bar{N}_r, \quad N_{r\theta} = \bar{N}_{r\theta}, \quad M_r = \bar{M}_r, \quad R_r = \bar{R}_r, \tag{II-5}$$

where

$$R_r = Q_r + \frac{1}{r} \frac{\partial M_{r\theta}}{\partial \theta} \tag{II-6}$$

and  $\bar{N}_r$ ,  $\bar{M}_r$ ,  $\bar{N}_{r\theta}$ , and  $\bar{R}_r$  are the applied edge loads and edge moment; and the displacement conditions

$$w = \bar{w} \quad , \quad u = \bar{u} \quad , \quad v = \bar{v} \quad , \quad w_{,r} = \bar{\beta}_r \quad , \tag{II-7}$$

where  $w$ ,  $u$ ,  $v$ , and  $\beta_r$  are the prescribed linear and angular displacements at the edge. In the subsequent development, other possible combinations are also considered.

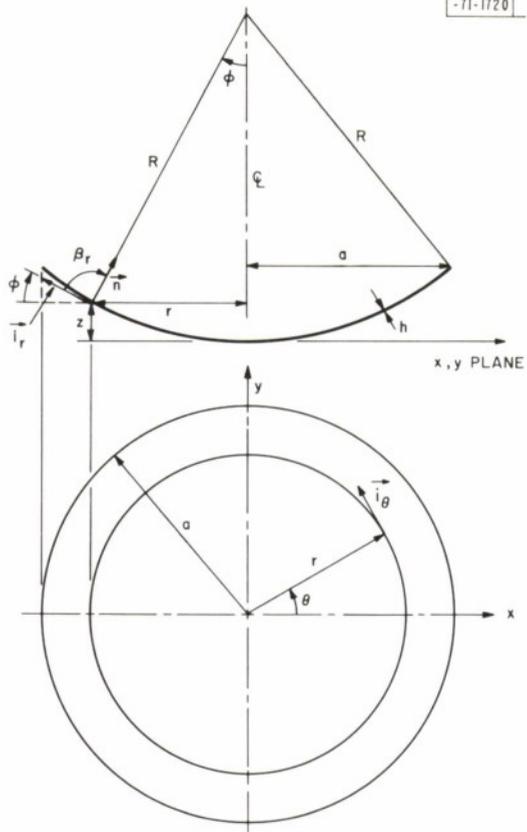


Fig. 1. Geometry of middle surface.

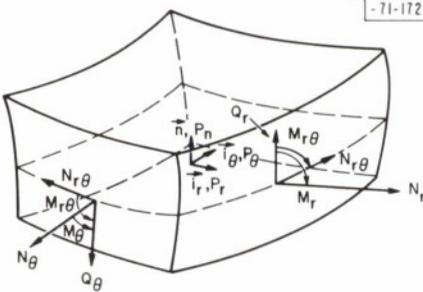


Fig. 2. Forces and moments.

### C. Solution to Homogeneous Differential Equations

The solution of the homogeneous system (II-4) is given by<sup>5</sup>

$$w_H = \psi + \chi \quad , \quad F_H = \varphi + RD\nabla^2\chi \quad , \quad (II-8)$$

where  $\varphi$  and  $\psi$  are harmonic functions, i.e.,

$$\nabla^2\varphi = \nabla^2\psi = 0 \quad , \quad (II-9)$$

and  $\chi$  satisfies the equation

$$\nabla^2\nabla^2\chi + \frac{1}{DAR^2}\chi = 0 \quad . \quad (II-10)$$

Since the shell is closed in the circumferential direction, the relevant displacement components, stress resultants, and moment resultants must be periodic functions of  $\Theta$  with period  $2n\pi$ ,  $n = 0, 1, 2, 3, \dots$ . Retaining only that portion of the solution which leads to the desired finiteness at the apex, we have

$$\begin{aligned} \varphi &= \sum_{n=0}^{\infty} r^n (A_n \cos n\Theta + D_n \sin n\Theta) \quad , \\ \psi &= \sum_{n=0}^{\infty} r^n (B_n \cos n\Theta + E_n \sin n\Theta) \quad , \end{aligned} \quad (II-11)$$

and

$$\chi = \sum_{n=0}^{\infty} [C_n J_n(kr) + H_n J_n(\bar{k}r)] [F_n \cos n\Theta + G_n \sin n\Theta] \quad ,$$

where

$$k = \frac{e^{i\pi/4}}{\sqrt[4]{DAR^2}} \quad , \quad \bar{k} = \frac{e^{-i\pi/4}}{\sqrt[4]{DAR^2}} \quad , \quad (II-12)$$

and where  $A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i$ , and  $K_i$  are arbitrary constants.  $J_n(x)$  is the Bessel function of the first kind and of order  $n$ . Properties of these functions can be found in Refs. 12 and 13.

For the present investigation, it suffices to consider only a special case of (II-11); namely,

$$\begin{aligned} \varphi &= A_n r^n \cos n\Theta \quad , \quad \psi = B_n r^n \cos n\Theta \quad , \\ \chi &= [H_n J_n(kr) + \bar{H}_n J_n(\bar{k}r)] \cos n\Theta \quad , \end{aligned} \quad (II-13)$$

where  $A_n$  and  $B_n$  are real constants;  $H_n$  and  $\bar{H}_n$  are complex constants with  $\bar{H}_n$  as the complex conjugate of  $H_n$  to ensure real solutions. In the subsequent development, we shall replace  $r$  by a dimensionless variable  $\rho$  defined by

$$\rho = \frac{r}{a} \quad (II-14)$$

so that  $0 \leq \rho \leq 1$  and take (II-13) in the form of

$$\begin{aligned} \varphi &= a^2 A_n \rho^n \cos n\theta \quad , \quad \psi = \frac{a^2 B_n}{D(1-\nu)} \rho^n \cos n\theta \\ \chi &= \frac{1}{\sqrt{D}} [C_n \text{ber}_n(\lambda\rho) + D_n \text{bei}_n(\lambda\rho)] \cos n\theta \quad , \end{aligned} \quad (\text{II-15})$$

where

$$\lambda = \frac{a}{\sqrt[4]{DAR^2}} \quad (\text{II-16})$$

$\text{ber}_n$  and  $\text{bei}_n$  are the  $n^{\text{th}}$  order Thomson functions and  $C_n$  and  $D_n$  are real constants.

For shallow shells with sufficiently small bending-to-stretching stiffness ratio  $DA$ , the dimensionless number  $\lambda$  is large compared with unity. For an isotropic homogeneous shell of constant wall thickness,

$$\lambda = \frac{a}{\sqrt{Rh}} \sqrt[4]{12(1-\nu^2)} \quad (\text{II-17})$$

and  $\lambda \rightarrow \infty$  as  $h \rightarrow 0$ . In the present work, we shall be concerned particularly with shells for which  $\lambda \gg 1$ . It is known (Ref. 5) that for this range of  $\lambda$ , the effect of  $\chi$  is confined to a narrow zone near the edge of the shell; it is therefore referred to as the edge-zone solution. Away from the edge,  $\varphi$  and  $\psi$  become dominant; together, they are referred to as the interior solution.

Corresponding to (II-15), we have [cf. (II-3), (II-4), and (II-6)]

$$\begin{aligned} F &= \left\{ a^2 A_n \rho^n - \frac{1}{\sqrt{A}} [C_n \text{bei}_n(\lambda\rho) - D_n \text{ber}_n(\lambda\rho)] \right\} \cos n\theta \\ w &= \left\{ \frac{a^2 B_n}{D(1-\nu)} \rho^n + \frac{1}{\sqrt{D}} [C_n \text{ber}_n(\lambda\rho) + D_n \text{bei}_n(\lambda\rho)] \right\} \cos n\theta \\ \frac{\partial w}{\partial r} &= \left\{ \frac{naB_n}{D(1-\nu)} \rho^{n-1} + \frac{\lambda}{a\sqrt{D}} [C_n \text{ber}'_n(\lambda\rho) + D_n \text{bei}'_n(\lambda\rho)] \right\} \cos n\theta \\ u &= \left\{ -na(1+\nu) AA_n \rho^{n-1} + \frac{a^3 B_n}{RD(n+1)(1-\nu)} \rho^{n+1} \right. \\ &\quad \left. + \frac{\lambda(1+\nu)\sqrt{A}}{a} [C_n \text{bei}'_n(\lambda\rho) - D_n \text{ber}'_n(\lambda\rho)] \right\} \cos n\theta \\ v &= \left\{ na(1+\nu) AA_n \rho^{n-1} + \frac{a^3 B_n}{RD(n+1)(1-\nu)} \rho^{n+1} \right. \\ &\quad \left. - \frac{n(1+\nu)\sqrt{A}}{a\rho} [C_n \text{bei}_n(\lambda\rho) - D_n \text{ber}_n(\lambda\rho)] \right\} \sin n\theta \\ Q_r &= \frac{\lambda}{aR\sqrt{A}} [C_n \text{bei}'_n(\lambda\rho) - D_n \text{ber}'_n(\lambda\rho)] \cos n\theta \\ Q_\theta &= -\frac{n}{R\sqrt{A}(a\rho)} [C_n \text{bei}_n(\lambda\rho) - D_n \text{ber}_n(\lambda\rho)] \sin n\theta \end{aligned}$$

$$\begin{aligned}
N_R &= - \left\{ n(n-1) A_n \rho^{n-2} + \frac{\lambda}{\sqrt{A} (a^2 \rho)} [C_n f_{rc}(\lambda \rho) - D_n f_{rd}(\lambda \rho)] \right\} \cos n\theta \\
N_\Theta &= \left\{ n(n-1) A_n \rho^{n-2} - \frac{\lambda^2}{\sqrt{A} a^2} [C_n f_{\Theta c}(\lambda \rho) + D_n f_{\Theta d}(\lambda \rho)] \right\} \cos n\theta \\
N_{r\Theta} &= \left\{ n(n-1) A_n \rho^{n-2} - \frac{n\lambda}{\sqrt{A} (a^2 \rho)} [C_n f_{sc}(\lambda \rho) - D_n f_{sd}(\lambda \rho)] \right\} \sin n\theta \\
M_R &= \left\{ -n(n-1) B_n \rho^{n-2} + \frac{1}{R \sqrt{A}} [C_n g_{rc}(\lambda \rho) - D_n g_{rd}(\lambda \rho)] \right\} \cos n\theta \\
M_\Theta &= \left\{ n(n-1) B_n \rho^{n-2} + \frac{1}{R \sqrt{A}} [C_n g_{\Theta c}(\lambda \rho) - D_n g_{\Theta d}(\lambda \rho)] \right\} \cos n\theta \\
M_{r\Theta} &= \left\{ n(n-1) B_n \rho^{n-2} + \frac{n(1-\nu)}{R \sqrt{A} (\lambda \rho)} [C_n g_{sc}(\lambda \rho) + D_n g_{sd}(\lambda \rho)] \right\} \sin n\theta \\
R_R &= \left\{ \frac{n^2(n-1) B_n}{a} \rho^{n-3} + \frac{\lambda}{aR \sqrt{A}} [C_n f_{nc}(\lambda \rho) - D_n f_{nd}(\lambda \rho)] \right\} \cos n\theta \quad , \quad (II-18)
\end{aligned}$$

where

$$\begin{aligned}
f_{rc}(x) &= \text{bei}'_n(x) - \frac{n^2}{x} \text{bei}_n(x) \\
f_{rd}(x) &= \text{ber}'_n(x) - \frac{n^2}{x} \text{ber}_n(x) \\
f_{\Theta c}(x) &= \text{ber}_n(x) - \frac{1}{x} \left[ \text{bei}'_n(x) - \frac{n^2}{x} \text{bei}_n(x) \right] \\
f_{\Theta d}(x) &= \text{bei}_n(x) + \frac{1}{x} \left[ \text{ber}'_n(x) - \frac{n^2}{x} \text{ber}_n(x) \right] \\
f_{sc}(x) &= \text{bei}'_n(x) - \frac{1}{x} \text{bei}_n(x) \\
f_{sd}(x) &= \text{ber}'_n(x) - \frac{1}{x} \text{ber}_n(x) \\
f_{nc}(x) &= \text{bei}'_n(x) + \frac{n^2(1-\nu)}{x^2} \left[ \text{ber}'_n(x) - \frac{1}{x} \text{ber}_n(x) \right] \\
f_{nd}(x) &= \text{ber}'_n(x) - \frac{n^2(1-\nu)}{x^2} \left[ \text{bei}'_n(x) - \frac{1}{x} \text{bei}_n(x) \right] \\
g_{rc}(x) &= \text{bei}_n(x) + \frac{1-\nu}{x} \left[ \text{ber}'_n(x) - \frac{n^2}{x} \text{ber}_n(x) \right] \\
g_{rd}(x) &= \text{ber}_n(x) - \frac{1-\nu}{x} \left[ \text{bei}'_n(x) - \frac{n^2}{x} \text{bei}_n(x) \right] \\
g_{\Theta c}(x) &= \nu \text{bei}_n(x) - \frac{1-\nu}{x} \left[ \text{ber}'_n(x) - \frac{n^2}{x} \text{ber}_n(x) \right]
\end{aligned}$$

$$g_{\Theta d}(x) = \nu \text{ber}_n(x) + \frac{1-\nu}{x} \left[ \text{bei}'_n(x) - \frac{n^2}{x} \text{bei}_n(x) \right]$$

$$g_{Sc}(x) = \text{ber}'_n(x) - \frac{1}{x} \text{ber}_n(x)$$

$$g_{Sd}(x) = \text{bei}'_n(x) - \frac{1}{x} \text{bei}_n(x) \quad .$$

### III. PROBLEM OF PRESCRIBED EDGE LOADS AND EDGE MOMENTS

#### A. Prescribed Conditions at Edge

In this section, we consider a shell without surface loads acted upon by the following system of edge loads and moments at  $\rho = 1$ :

$$\begin{aligned} \bar{N}_r &= N_n \cos n\theta \quad , \quad \bar{N}_{r\theta} = S_n \sin n\theta \quad , \\ \bar{M}_r &= M_n \cos n\theta \quad , \quad \bar{R}_r = R_n \cos n\theta \quad , \end{aligned} \quad (\text{III-1})$$

where the fixed integer  $n$  is greater than unity. A suitable solution to this problem is that given by (II-18). The boundary conditions (II-5) become

$$\begin{aligned} n(n-1) A_n + \frac{\lambda}{a^2 \sqrt{A}} [C_n f_{rc}(\lambda) - D_n f_{rd}(\lambda)] &= -N_n \\ n(n-1) A_n - \frac{n\lambda}{a^2 \sqrt{A}} [C_n f_{sc}(\lambda) - D_n f_{sd}(\lambda)] &= S_n \\ -n(n-1) B_n + \frac{1}{R \sqrt{A}} [C_n g_{rc}(\lambda) - D_n g_{rd}(\lambda)] &= M_n \\ n^2(n-1) B_n + \frac{\lambda}{R \sqrt{A}} [C_n f_{nc}(\lambda) - D_n f_{nd}(\lambda)] &= aR_n \quad . \end{aligned} \quad (\text{III-2})$$

Solving (III-2) for  $A_n$  and  $B_n$ , we get

$$A_n = \frac{1}{2n(n-1)} [S_n X_1 - N_n X_2 + \frac{n-1}{\alpha a} (aR_n + nM_n) X_3] \quad (\text{III-3})$$

$$B_n = \frac{\alpha a}{2n^2(n^2-1)} [(S_n + N_n) X_4 + \frac{n+1}{\alpha a} (aR_n X_1 - nM_n X_2)] \quad , \quad (\text{III-4})$$

where

$$\begin{aligned} \alpha &= \frac{a}{R} \quad (\text{III-5}) \\ X_1 &= \frac{1}{\Delta_1} \left\{ 1 - \frac{1-\nu}{\lambda} \left[ \alpha_2 - \frac{n(n+1)}{\lambda} \alpha_3 + \frac{n^3}{\lambda^2} \alpha_4 \right] \right\} \\ X_2 &= \frac{1}{\Delta_1} \left\{ 1 - \frac{n(1-\nu)}{\lambda} \left[ \alpha_2 - \frac{n+1}{\lambda} \alpha_3 + \frac{n}{\lambda^2} \alpha_4 \right] \right\} \\ X_3 &= \frac{1}{\Delta_1} \quad . \end{aligned}$$

$$X_4 = \frac{1}{\Delta_1} \left\{ 1 - \frac{1-\nu}{\lambda} \left[ \alpha_2 - \frac{2n^2}{\lambda} \alpha_3 + \frac{n^2}{\lambda^2} \alpha_4 + \frac{n^2(n^2-1)(1-\nu)}{\lambda^3} \right] \right\}, \quad (\text{III-6})$$

with

$$\Delta_1 = 1 - \frac{(n+1)(1-\nu)}{2\lambda} \left( \alpha_2 - \frac{2n}{\lambda} \alpha_3 + \frac{n^2}{\lambda^2} \alpha_4 \right) \quad (\text{III-7})$$

$$\alpha_i = \frac{\beta_i}{\beta_1} \quad (i = 2, 3, 4) \quad (\text{III-8})$$

$$\begin{aligned} \beta_1 &= \text{ber}_n(\lambda) \text{bei}'_n(\lambda) - \text{bei}_n(\lambda) \text{ber}'_n(\lambda) \\ \beta_2 &= [\text{ber}'_n(\lambda)]^2 + [\text{bei}'_n(\lambda)]^2 \\ \beta_3 &= \text{ber}_n(\lambda) \text{ber}'_n(\lambda) + \text{bei}_n(\lambda) \text{bei}'_n(\lambda) \\ \beta_4 &= \text{ber}_n^2(\lambda) + \text{bei}_n^2(\lambda) \end{aligned} \quad (\text{III-9})$$

It is clear from Fig. 1 that  $\alpha$  is the slope of the middle surface at the edge of the shell relative to the base plane. By the definition of a shallow shell (Ref. 7),  $\alpha^2 \ll 1$  so that

$$1 + \alpha^2 \cong 1 \quad (\text{III-10})$$

### B. Some Relevant Asymptotic Relations

For large values of  $\lambda$ , we have the following asymptotic expansions for  $\text{ber}_n(\lambda)$ ,  $\text{bei}_n(\lambda)$ ,  $\text{ber}'_n(\lambda)$ , and  $\text{bei}'_n(\lambda)$  (Ref. 12):

$$\begin{aligned} \text{ber}_n(\lambda) &\sim \frac{e^{\lambda/\sqrt{2}}}{\sqrt{2\pi\lambda}} \left[ \cos\left(\frac{\lambda}{\sqrt{2}} - \frac{\pi}{8} + \frac{n\pi}{2}\right) - \frac{(4n^2-1)}{8\lambda} \cos\left(\frac{\lambda}{\sqrt{2}} - \frac{3\pi}{8} + \frac{n\pi}{2}\right) + O\left(\frac{1}{\lambda^2}\right) \right] \\ \text{bei}_n(\lambda) &\sim \frac{e^{\lambda/\sqrt{2}}}{\sqrt{2\pi\lambda}} \left[ \sin\left(\frac{\lambda}{\sqrt{2}} - \frac{\pi}{8} + \frac{n\pi}{2}\right) - \frac{(4n^2-1)}{8\lambda} \sin\left(\frac{\lambda}{\sqrt{2}} - \frac{3\pi}{8} + \frac{n\pi}{2}\right) + O\left(\frac{1}{\lambda^2}\right) \right] \\ \text{ber}'_n(\lambda) &\sim \frac{e^{\lambda/\sqrt{2}}}{\sqrt{2\pi\lambda}} \left[ \cos\left(\frac{\lambda}{\sqrt{2}} + \frac{\pi}{8} + \frac{n\pi}{2}\right) - \frac{4n^2+3}{8\lambda} \cos\left(\frac{\lambda}{\sqrt{2}} - \frac{\pi}{8} + \frac{n\pi}{2}\right) + O\left(\frac{1}{\lambda^2}\right) \right] \\ \text{bei}'_n(\lambda) &\sim \frac{e^{\lambda/\sqrt{2}}}{\sqrt{2\pi\lambda}} \left[ \sin\left(\frac{\lambda}{\sqrt{2}} + \frac{\pi}{8} + \frac{n\pi}{2}\right) - \frac{4n^2+3}{8\lambda} \sin\left(\frac{\lambda}{\sqrt{2}} - \frac{\pi}{8} + \frac{n\pi}{2}\right) + O\left(\frac{1}{\lambda^2}\right) \right] \end{aligned} \quad (\text{III-11})$$

From these, we get

$$\alpha_2 \sim \sqrt{2} \left[ 1 - \frac{1}{\sqrt{2}\lambda} + O\left(\frac{1}{\lambda^2}\right) \right]$$

$$\begin{aligned}\alpha_3 &\sim \left[ 1 - \frac{1}{\sqrt{2}\lambda} + O\left(\frac{1}{\lambda^2}\right) \right] \\ \alpha_4 &\sim \sqrt{2} \left[ 1 + O\left(\frac{1}{\lambda^2}\right) \right] .\end{aligned}\tag{III-12}$$

We shall have occasions to use these results in the subsequent development. A new parameter  $\mu$  is defined at this point to take the place of  $\lambda$  in all subsequent asymptotic considerations.

$$\mu = \frac{\lambda}{\sqrt{2}} .\tag{III-13}$$

### C. Asymptotic Interior Solution

For  $\mu \gg n > 1$ , the following asymptotic expressions for the  $X_i$ 's may be obtained with the help of (III-12):

$$\begin{aligned}X_1 &\sim 1 + \frac{(n-1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) , & X_2 &\sim 1 - \frac{(n-1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) , \\ X_3 &\sim 1 + \frac{(n+1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) , & X_4 &\sim 1 + \frac{(n-1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) .\end{aligned}\tag{III-14}$$

Correspondingly, we have for  $\mu \gg n > 1$

$$\begin{aligned}A_n &\sim \frac{1}{2n(n-1)} \left\{ \left[ 1 + \frac{(n-1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] S_n - \left[ 1 - \frac{(n-1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] N_n \right. \\ &\quad \left. + \frac{n-1}{\alpha a} \left[ 1 + \frac{(n+1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] (aR_n + nM_n) \right\} ,\end{aligned}\tag{III-15}$$

$$\begin{aligned}B_n &\sim \frac{\alpha a}{2n^2(n^2-1)} \left\{ \left[ 1 + \frac{(n-1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] (S_n + N_n) + \frac{n+1}{\alpha a} \right. \\ &\quad \left. \times \left[ 1 + \frac{(n-1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] aR_n - \frac{n+1}{\alpha a} \left[ 1 - \frac{(n-1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] nM_n \right\} .\end{aligned}\tag{III-16}$$

If only the leading terms of the expansions for the  $X_i$ 's are retained, (III-15) and (III-16) are further reduced to

$$A_n \sim \frac{1}{2n(n-1)} [(S_n - N_n) + \frac{n-1}{\alpha a} (aR_n + nM_n)] ,\tag{III-17}$$

$$B_n \sim \frac{\alpha a}{2n^2(n^2-1)} [(S_n + N_n) + \frac{n+1}{\alpha a} (aR_n - nM_n)] .\tag{III-18}$$

Equations (III-17) and (III-18) are exactly those obtained by Reissner<sup>4</sup> for the same problem (the discrepancy in the sign associated with  $\alpha$  is due to choice of the direction of the positive middle surface normal). Equations (III-15) and (III-16) show that (III-17) and (III-18) as well as their consequences are significant only if

$$\frac{1}{\mu} | (S_n + N_n) + \frac{n+1}{\alpha a} (aR_n + nM_n) | \ll | (S_n - N_n) + \frac{n-1}{\alpha a} (aR_n + nM_n) | \tag{III-19}$$

and

$$\frac{1}{\mu} | (S_n + N_n) + \frac{n+1}{\alpha a} (aR_n + nM_n) | \ll | (S_n + N_n) + \frac{n+1}{\alpha a} (aR_n - nM_n) | \tag{III-20}$$

#### D. Interior Membrane and Inextensional Bending Stresses

To examine the relative importance of the membrane and inextensional bending stresses in the interior of the shell, it suffices to consider two representative quantities  $\sigma_{\Theta D}$  and  $\sigma_{rB}$  given by

$$\sigma_{\Theta D} = \frac{N_{\Theta}}{h} = \frac{n(n-1)}{h} A_n \rho^{n-2} \cos n\theta$$

$$\sigma_{rB} = \frac{6M_r}{h^2} = -\frac{6n(n-1)}{h^2} B_n \rho^{n-2} \cos n\theta$$

From these, we get

$$\frac{\sigma_D}{\sigma_B} = O\left(\frac{hA_n}{B_n}\right) \quad (III-21)$$

For  $\mu \gg n > 1$ , we have

$$\begin{aligned} \frac{hA_n}{B_n} = \frac{n(n+1)h}{\alpha a} & \left\{ \left[ 1 + O\left(\frac{1}{\mu^2}\right) \right] (S_n - N_n) + \frac{(n-1)(1-\nu)}{2\mu} \left[ 1 + O\left(\frac{1}{\mu}\right) \right] (S_n + N_n) \right. \\ & \left. + \frac{n-1}{\alpha a} \left[ 1 + O\left(\frac{1}{\mu}\right) \right] (aR_n + nM_n) \right\} / \left\{ \left[ 1 + O\left(\frac{1}{\mu^2}\right) \right] (S_n + N_n) + \frac{n+1}{\alpha a} \left[ 1 + O\left(\frac{1}{\mu^2}\right) \right] \right. \\ & \left. \times (aR_n - nM_n) + \frac{(n^2-1)(1-\nu)}{2\mu} \left[ 1 + O\left(\frac{1}{\mu}\right) \right] (aR_n + nM_n) \right\} \quad (III-22) \end{aligned}$$

For a given set of  $N_n$ ,  $S_n$ ,  $R_n$ , and  $M_n$ , the relative order of magnitude of  $\sigma_B$  and  $\sigma_D$  can be obtained by way of (III-21) and (III-22). In what follows, we shall restrict our attention to an isotropic and homogenous shell so that [cf. (III-5) and (II-17)]

$$\frac{\alpha a}{h} = \frac{a^2}{Rh} = \frac{\mu^2}{\sqrt{3(1-\nu^2)}} \quad (III-23)$$

For the class of problems for which  $R_n = M_n = O$ , (III-22) becomes

$$\begin{aligned} \frac{hA_n}{B_n} = \frac{n(n+1)\sqrt{3(1-\nu^2)}}{\mu^2} \\ \times \left\{ \frac{\left[ 1 + O\left(\frac{1}{\mu^2}\right) \right] [S_n - N_n] + \frac{(n-1)(1-\nu)}{2\mu} \left[ 1 + O\left(\frac{1}{\mu}\right) \right] [S_n + N_n]}{\left[ 1 + O\left(\frac{1}{\mu}\right) \right] [S_n + N_n]} \right\} \quad (III-24) \end{aligned}$$

Clearly, the nature of the interior stress state depends on the relative magnitude of  $(S_n + N_n)$  and  $(S_n - N_n)$ . We now consider the various relative magnitudes and their consequences.

- (1) If  $|S_n - N_n| \ll |S_n + N_n|/\mu$ , we may write (except for higher-order terms) (III-24) as

$$\frac{hA_n}{B_n} = \frac{n(n^2-1)(1-\nu)\sqrt{3(1-\nu^2)}}{2\mu^3} \quad (III-25)$$

Therewith,

$$\frac{\sigma_D}{\sigma_B} = O\left(\frac{1}{\mu^3}\right) \quad . \quad (\text{III-26})$$

The membrane stresses are therefore small compared with the bending stresses in the interior of the shell.

- (2) If  $|S_n - N_n|$  and  $|S_n + N_n|/\mu$  are of the same order of magnitude, both terms in the numerator of (III-24) are equally important. Nevertheless, we may still conclude that

$$\frac{\sigma_D}{\sigma_B} = O\left(\frac{hA_n}{B_n}\right) = O\left(\frac{1}{\mu^3}\right) \quad (\text{III-27})$$

which is the same as (1).

- (3) If  $(S_n - N_n)$  and  $(S_n + N_n)$  are of the same order of magnitude, we may write (III-24) as

$$\frac{hA_n}{B_n} = \frac{n(n+1)\sqrt{3(1-\nu^2)}}{\mu^2} \left( \frac{S_n - N_n}{S_n + N_n} \right) \quad (\text{III-28})$$

or

$$\frac{\sigma_D}{\sigma_B} = O\left(\frac{1}{\mu^2}\right) \quad . \quad (\text{III-29})$$

Although the bending stresses still dominate in the interior of the shell, the relative order of magnitude between membrane and bending stresses is not the same as that of (1) and (2).

- (4) If  $(S_n + N_n) = O(|S_n - N_n|/\mu)$ , (III-28) continues to hold. But now we have, instead of (III-29),

$$\frac{\sigma_D}{\sigma_B} = O\left(\frac{1}{\mu}\right) \quad . \quad (\text{III-30})$$

- (5) If  $(S_n + N_n) = O(|S_n - N_n|/\mu^2)$ ,

$$\frac{\sigma_D}{\sigma_B} = O(1) \quad . \quad (\text{III-31})$$

The interior of the shell is therefore in a mixed stress state; that is, both the membrane and bending stresses are equally important.

- (6) Finally, if  $(S_n + N_n) \ll (S_n - N_n)/\mu^2$ ,

$$\frac{\sigma_D}{\sigma_B} \gg 1 \quad . \quad (\text{III-32})$$

The interior stresses are predominantly membrane stresses. In particular, if  $S_n + N_n = 0$  (keeping in mind that  $R_n = M_n = 0$ ), there would be no inextensional bending action in the shell interior [see (III-3) and (III-4)]. On the other hand, it should be noted that it is not possible to have no membrane action for  $R_n = M_n = 0$ .

For the class of problems for which  $S_n = N_n = 0$ , (III-22) becomes

$$\frac{hA_n}{B_n} = \frac{n(n-1)\sqrt{3(1-\nu^2)}}{\mu^2} \times \left\{ \frac{[1 + O(\frac{1}{\mu})] [aR_n + nM_n]}{\left[1 + O\left(\frac{1}{\mu}\right)\right] (aR_n - nM_n) + \frac{(n-1)(1-\nu)}{2\mu} [1 + O(\frac{1}{\mu})] (aR_n + nM_n)} \right\} \quad (\text{III-33})$$

- (1) If  $|aR_n + nM_n|/\mu \ll |aR_n - nM_n|$ , (III-33) reduces, except for higher-order terms, to

$$\frac{hA_n}{B_n} = \frac{n(n-1)\sqrt{3(1-\nu^2)}}{\mu^2} \left\{ \frac{aR_n + nM_n}{aR_n - nM_n} \right\} \quad (\text{III-34})$$

Therewith,

$$\frac{\sigma_D}{\sigma_B} \ll \frac{1}{\mu} \quad (\text{III-35})$$

In particular, if  $aR_n + nM_n = 0$  (keeping in mind that  $S_n = N_n = 0$ ), there would be no membrane action in the interior of the shell [see (III-3) and (III-4)].

- (2) If  $(aR_n + nM_n)/\mu$  and  $(aR_n - nM_n)$  are of the same order of magnitude, both terms in the denominator are equally important. If their sum is again of the same order of magnitude as each of the terms, then

$$\frac{\sigma_D}{\sigma_B} = O\left(\frac{1}{\mu}\right) \quad (\text{III-36})$$

Otherwise, we must investigate the higher-order terms in the asymptotic expansions for  $X_1$  and  $X_2$  to determine whether the interior of the shell is in a mixed stress state or a membrane stress state.

- (3) If  $|aR_n - nM_n| \ll |aR_n + nM_n|/\mu$ , except for higher-order terms, we have

$$\frac{hA_n}{B_n} = \frac{2n(n-1)\sqrt{3(1-\nu^2)}}{\mu} \quad (\text{III-37})$$

Therewith,

$$\frac{\sigma_D}{\sigma_B} = O\left(\frac{1}{\mu}\right) \quad (\text{III-38})$$

Thus, for the class of problems in which the contribution of  $S_n$  and  $N_n$  to  $A_n$  and  $B_n$  are negligible, the interior of the shell is almost always in a state of inextensional bending. The only exception occurs when  $(aR_n + nM_n)/\mu$  and  $(aR_n - nM_n)$  are of the same order of magnitude and

$$\left| (aR_n - nM_n) + \frac{(n-1)(1-\nu)}{2\mu} (aR_n + nM_n) \right| \ll |aR_n - nM_n| \quad (\text{III-39})$$

For this exception, the relevant third-order terms of the asymptotic expression for  $B_n$  must be considered in order to determine whether the interior is in a mixed state or a membrane state of stress. In short, we may say that, while self-equilibrating tangential edge loads alone may lead to any one of the three possible interior stress states, it takes a rather special

combination of transverse load and moment (without tangential edge loads) to induce an interior state other than an inextensional bending state.

### E. Membrane and Inextensional Bending Solution

If only the membrane and inextensional bending components of the solution are used to determine the interior state of the shell without explicit reference to the edge-zone solution, the number of constants of integration available for the satisfaction of prescribed conditions at the edge is reduced to two:  $A_n$  and  $B_n$ . Correspondingly, a reduction of the total number of boundary conditions to be satisfied is necessary. It has been shown that the appropriately contracted stress boundary conditions at the edge  $\rho = 1$  are<sup>10, 11</sup>

$$\begin{aligned} (N_r - \bar{N}_r) + \frac{1}{\alpha} (R_r - \bar{R}_r) + \frac{1}{\alpha a} (M_r - \bar{M}_r),_{\theta\theta} &= 0 \quad , \\ (N_{r\theta} - \bar{N}_{r\theta}) - \frac{1}{\alpha} (R_r - \bar{R}_r),_{\theta} + \frac{1}{\alpha a} (M_r - \bar{M}_r),_{\theta} &= 0 \quad , \end{aligned} \quad \text{(III-40)}$$

where the quantities  $N_r$ ,  $N_{r\theta}$ ,  $M_r$ , and  $R_r$  must now be taken as the sum of the solutions obtained by a membrane consideration and those by an inextensional bending consideration.

For the present problem, (III-40) leads to the following expressions for the constants  $A_n$  and  $B_n$ :

$$A_n = \frac{1}{2n(n-1)} [(S_n - N_n) + \frac{n-1}{\alpha a} (aR_n + nM_n)] \quad , \quad \text{(III-41)}$$

$$B_n = \frac{\alpha a}{2n^2(n^2-1)} [(S_n + N_n) + \frac{n+1}{\alpha a} (aR_n - nM_n)] \quad . \quad \text{(III-42)}$$

A comparison of (III-41) and (III-42) with (III-17) and (III-18) shows that the interior solution obtained in this way is just the leading term of the asymptotic expansion of the exact interior solution. In view of the discussion of Sec. III-D, this simple interior solution may not be an adequate first approximation of the exact interior solution, if (III-19) and (III-20) are not satisfied.

### F. Influence Coefficients

The deformation at the edge of the shell will now be expressed in terms of the applied edge loads and edge moments. From (II-18), we have, for  $\rho = 1$  (or  $r = a$ ),

$$\begin{aligned} w_n &= \frac{a^2 B_n}{D(1-\nu)} + \frac{1}{\sqrt{D}} [C_n \text{ber}_n(\lambda) + D_n \text{bei}_n(\lambda)] \\ \beta_n &= \frac{na B_n}{D(1-\nu)} + \frac{\lambda}{a\sqrt{D}} [C_n \text{ber}'_n(\lambda) + D_n \text{bei}'_n(\lambda)] \\ u_n &= -na(1+\nu) AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} + \frac{(1+\nu)\lambda\sqrt{A}}{a} [C_n \text{bei}'_n(\lambda) - D_n \text{ber}'_n(\lambda)] \\ v_n &= na(1+\nu) AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} - \frac{n(1+\nu)\sqrt{A}}{a} [C_n \text{bei}_n(\lambda) - D_n \text{ber}_n(\lambda)] \quad . \end{aligned} \quad \text{(III-43)}$$

where  $w_n$ ,  $u_n$ , and  $v_n$  are the three edge displacement components and  $\beta_n$  is the meridional slope change at the shell edge. In order to get the influence coefficients, we need  $C_n$  and  $D_n$  in addition to  $A_n$  and  $B_n$ . To compute  $C_n$  and  $D_n$ , we observe that  $A_n$  and  $B_n$  can be eliminated from (III-2) to yield

$$\begin{aligned} \frac{\lambda}{a^2 \sqrt{A}} \{C_n [f_{rc}(\lambda) + n f_{sc}(\lambda)] - D_n [f_{rd}(\lambda) + n f_{sd}(\lambda)]\} &= -(N_n + S_n) \\ \frac{1}{R \sqrt{A}} \{C_n [n g_{rc}(\lambda) + \lambda f_{nc}(\lambda)] - D_n [n g_{rd}(\lambda) + \lambda f_{nd}(\lambda)]\} &= (n M_n + a R_n) \end{aligned} \quad (III-44)$$

Solving (III-44) for  $C_n$  and  $D_n$ , we get

$$C_n = -\frac{R \sqrt{A}}{\Delta_1} [(a R_n + n M_n) (f_{rd} + n f_{sd}) + \alpha a (S_n + N_n) (f_{nd} + \frac{n}{\lambda} g_{rd})] \quad (III-45)$$

$$D_n = -\frac{R \sqrt{A}}{\Delta_1} [(a R_n + n M_n) (f_{rc} + n f_{sc}) - \alpha a (S_n + N_n) (f_{nc} + \frac{n}{\lambda} g_{rc})] \quad (III-46)$$

where  $\Delta_1$  was defined by (III-7). Substituting (III-3), (III-4), (III-45), and (III-46) into (III-43), we get the following matrix equation

$$\begin{Bmatrix} w_n \\ \beta_n \\ u_n \\ v_n \end{Bmatrix} = \begin{bmatrix} K_{wR} & K_{wM} & K_{wN} & K_{wS} \\ K_{\beta R} & K_{\beta M} & K_{\beta N} & K_{\beta S} \\ K_{uR} & K_{uM} & K_{uN} & K_{uS} \\ K_{vR} & K_{vM} & K_{vN} & K_{vS} \end{bmatrix} \begin{Bmatrix} R_n \\ M_n \\ N_n \\ S_n \end{Bmatrix} \quad (III-47)$$

where the elements of the influence coefficient matrix are

$$\begin{aligned} K_{wR} &= \left(\frac{a}{n}\right)^2 K_{\beta M} = \frac{t}{\Delta_1} \left[ 1 - \frac{1-\nu}{\lambda} \left( \alpha_2 - \frac{2n}{\lambda} \alpha_3 + \frac{n^2}{\lambda^2} \alpha_4 \right) \right] \\ K_{wM} = K_{\beta R} &= \left(\frac{n+1}{\alpha}\right) K_{\beta N} = \left(\frac{n+1}{\alpha}\right) K_{\beta S} = \left(\frac{n+1}{\alpha}\right) K_{uM} = \left(\frac{n+1}{\alpha}\right) K_{vM} \\ &= \frac{n}{a} \frac{t}{\Delta_1} \left[ 1 - \frac{n(1-\nu)}{\lambda} \left( \alpha_2 - \frac{n+1}{\alpha} \alpha_3 + \frac{n}{\lambda^2} \alpha_4 \right) \right] \\ K_{wN} = K_{wS} = K_{uR} = K_{vR} &= \frac{\alpha}{n+1} \frac{t}{\Delta_1} \left\{ 1 - \frac{1-\nu}{\lambda} \left[ \alpha_2 - \frac{n(n+1)}{\lambda} \alpha_3 + \frac{n^3}{\lambda^2} \alpha_4 \right] \right\} \\ K_{uN} = K_{vS} &= \left(\frac{\alpha}{n+1}\right)^2 \frac{t}{\Delta_1} \left\{ \left[ 1 + \frac{2n^2(n+1)(1-\nu^2) - n^2(n^2-1)(1-\nu)^2}{\lambda^4} \right] \right. \\ &\quad \left. - \frac{1-\nu}{\lambda} \left( \alpha_2 - \frac{2n^2}{\lambda} \alpha_3 + \frac{n^2}{\lambda^2} \alpha_4 \right) \left[ 1 - \frac{n^2(n+1)^2(1-\nu^2)}{\lambda^4} \right] \right\} \end{aligned}$$

$$\begin{aligned}
K_{uS} = K_{vN} = & \left(\frac{\alpha}{n+1}\right)^2 \frac{t}{\Delta_1} \left\{ \left[ 1 - \frac{n^2(n^2-1)(1-\nu)^2 + 2n^3(n+1)(1-\nu^2)}{\lambda^4} \right] \right. \\
& - \left(\frac{1-\nu}{\lambda}\right) \alpha_2 \left[ 1 - \frac{n^3(n+1)^2(1-\nu^2)}{\lambda^4} \right] + \frac{2n^2(1-\nu)}{\lambda^2} \alpha_3 \left[ 1 - \frac{n(n+1)^2(1-\nu)}{\lambda^4} \right] \\
& \left. + \frac{n^2(1-\nu)}{\lambda^3} \alpha_3 \left[ 1 - \frac{n^3(n+1)^2(1-\nu^2)}{\lambda^4} \right] \right\}
\end{aligned}$$

with

$$t = \frac{a^3}{2n^2(n-1)(1-\nu)D} \quad (III-48)$$

For  $\mu \gg n > 1$ , we have the following asymptotic expressions for the flexibility matrix K:

$$K \sim t \begin{bmatrix} K_p & \frac{n}{a} K_m & \left(\frac{\alpha}{n+1}\right) K_p & \left(\frac{\alpha}{n+1}\right) K_p \\ \frac{n}{a} K_m & \left(\frac{n}{a}\right)^2 K_p & \frac{n\alpha}{(n+1)a} K_m & \frac{n\alpha}{(n+1)a} K_m \\ \frac{\alpha}{n+1} K_p & \frac{n\alpha}{(n+1)a} K_m & \left(\frac{\alpha}{n+1}\right)^2 K_p & \left(\frac{\alpha}{n+1}\right)^2 K_p \\ \frac{\alpha}{n+1} K_p & \frac{n\alpha}{(n+1)a} K_m & \left(\frac{\alpha}{n+1}\right)^2 K_p & \left(\frac{\alpha}{n+1}\right) K_p \end{bmatrix} + O\left(\frac{t}{\mu}\right)$$

where

$$\begin{aligned}
K_p &= 1 + \frac{(n-1)(1-\nu)}{2\mu} \\
K_m &= 1 - \frac{(n-1)(1-\nu)}{2\mu} \quad (III-49)
\end{aligned}$$

It should be pointed out that the last two columns as well as the last two rows of the matrix above are the same. So, we must consider either the higher-order terms in (III-49) or return to (III-47) if we want to invert K to get the stiffness matrix.

#### IV. PROBLEM OF PRESCRIBED EDGE DISPLACEMENTS

##### A. Prescribed Conditions at Edge

In this section we shall again consider a shell without surface loads. The shell is subjected to a system of edge loads and moments at  $\rho = 1$  to produce the following edge deformations:

$$\begin{aligned}
\bar{v} &= v_n \sin n\theta \quad , \quad \bar{u} = u_n \cos n\theta \quad , \\
\bar{w} &= w_n \cos n\theta \quad , \quad \bar{\beta}_r = \beta_n \cos n\theta \quad , \quad (IV-1)
\end{aligned}$$

where  $n \geq 2$ . Again the solution to this problem is given by (II-18). The boundary conditions (II-7) become:

$$\frac{a^2 B_n}{D(1-\nu)} + \frac{1}{\sqrt{D}} [C_n \text{ber}_n(\lambda) + D_n \text{bei}_n(\lambda)] = w_n$$

$$\frac{na^2 B_n}{D(1-\nu)} + \frac{\lambda}{\sqrt{D}} [C_n \text{ber}'_n(\lambda) + D_n \text{bei}'_n(\lambda)] = a\beta_n$$

$$na(1+\nu) AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} - \frac{n(1+\nu)\sqrt{A}}{a} [C_n \text{bei}_n(\lambda) - D_n \text{ber}_n(\lambda)] = v_n$$

$$-na(1+\nu) AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} + \frac{(1+\nu)\lambda\sqrt{A}}{a} [C_n \text{bei}'_n(\lambda) - D_n \text{ber}'_n(\lambda)] = u_n \quad . \quad (\text{IV-2})$$

Solving for  $A_n$  and  $B_n$ , we get

$$A_n = \frac{-1}{2naA(1+\nu)} \{u_n Y_1 - v_n Y_2 - 2\alpha(1+\nu) [w_n Y_3 - a\beta_n Y_4]\} \quad (\text{IV-3})$$

$$B_n = \frac{D(n+1)(1-\nu)}{2\alpha a^2} \{(u_n + v_n) Y_5 - 2\alpha(1+\nu) [w_n Y_6 - a\beta_n Y_7]\} \quad (\text{IV-4})$$

where

$$Y_1 = \frac{1}{\Delta_2} \left\{1 + \frac{n(n+1)(1+\nu)}{\lambda^2} [\alpha_3 - \frac{n}{\lambda} \alpha_4]\right\}$$

$$Y_2 = \frac{1}{\Delta_2} \left\{1 - \frac{(n+1)(1+\nu)}{\lambda} [\alpha_2 - \frac{n}{\lambda} \alpha_3]\right\}$$

$$Y_3 = \frac{1}{\lambda\Delta_2} \left[\alpha_2 + \frac{n}{\lambda} \alpha_3 - \frac{n^2(n+1)(1+\nu)}{\lambda^3}\right]$$

$$Y_4 = \frac{1}{\lambda^2\Delta_2} \left[\alpha_3 + \frac{n}{\lambda} \alpha_4 - \frac{n(n+1)(1+\nu)}{\lambda^2}\right]$$

$$Y_5 = \frac{1}{\Delta_2}$$

$$Y_6 = \frac{1}{2\lambda\Delta_2} (\alpha_2 - \frac{n}{\lambda} \alpha_3)$$

$$Y_7 = \frac{1}{2\lambda^2\Delta_2} (\alpha_3 - \frac{n}{\lambda} \alpha_4) \quad (\text{IV-5})$$

with

$$\Delta_2 = 1 - \frac{(n+1)(1+\nu)}{2\lambda} \left(\alpha_2 - \frac{2n}{\lambda} \alpha_3 + \frac{n^2}{\lambda^2} \alpha_4\right) \quad . \quad (\text{IV-6})$$

The quantities  $\alpha_i$  and  $\lambda$  are defined as in Secs. (III-A) and (II-B), respectively.

## B. Asymptotic Interior Solution

For  $\mu \gg n > 1$ , the following asymptotic expressions for the  $Y_i$ 's can be obtained with the help of (III-12) and (III-13):

$$Y_1 \sim 1 + \frac{(n+1)(1+\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right)$$

$$\begin{aligned}
Y_2 &\sim 1 - \frac{(n+1)(1+\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \\
Y_3 &\sim \frac{1}{2\mu} \left\{ 1 + \frac{1}{2\mu} [(n-1) + (n+1)(1+\nu)] + O\left(\frac{1}{\mu^2}\right) \right\} \\
Y_4 &\sim \frac{1}{(2\mu)^2} \left\{ 1 + \frac{1}{2\mu} [(2n-1) + (n+1)(1+\nu)] + O\left(\frac{1}{\mu^2}\right) \right\} \\
Y_5 &\sim 1 + \frac{(n+1)(1+\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \\
Y_6 &\sim \frac{1}{2\mu} \left[ 1 + \frac{\nu(n+1)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \\
Y_7 &\sim \frac{1}{(2\mu)^2} \left\{ 1 - \frac{1}{2\mu} [(2n+1) - (n+1)(1+\nu)] + O\left(\frac{1}{\mu^2}\right) \right\} \quad . \quad (IV-7)
\end{aligned}$$

Correspondingly, we have,

$$\begin{aligned}
A_n &\sim \frac{1}{2naA(1+\nu)} \left\{ \left[ 1 - \frac{(n+1)(1+\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] v_n - \left[ 1 + \frac{(n+1)(1+\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] u_n \right. \\
&\quad + \frac{2\alpha(1+\nu)w_n}{2\mu} \left[ 1 + \frac{(n-1) + (n+1)(1+\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \\
&\quad \left. - \frac{2\alpha(1+\nu)a\beta_n}{(2\mu)^2} \left[ 1 + \frac{(2n-1) + (n+1)(1+\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \right\} \quad (IV-8)
\end{aligned}$$

$$\begin{aligned}
B_n &\sim \frac{D(1-\nu)(n+1)}{2\alpha a^2} \left\{ \left[ 1 + \frac{(n+1)(1+\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] (u_n + v_n) \right. \\
&\quad - \frac{2\alpha(1+\nu)w_n}{2\mu} \left[ 1 + \frac{(n+1)\nu}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \\
&\quad \left. + \frac{2\alpha(1+\nu)a\beta_n}{(2\mu)^2} \left[ 1 + \frac{(n+1)(1+\nu) - (2n+1)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \right\} \quad . \quad (IV-9)
\end{aligned}$$

If the contribution from terms associated with  $w_n$  and  $\beta_n$  can be neglected and if terms of the order  $1/\mu$  in the remaining expression for  $A_n$  and  $B_n$  are also discarded, (IV-8) and (IV-9) are further reduced to

$$A_n \sim - \frac{(u_n - v_n)}{2na(1+\nu)A} \quad (IV-10)$$

$$B_n \sim \frac{D(1-\nu)(n+1)}{2\alpha a^2} (u_n + v_n) \quad (IV-11)$$

respectively. Equations (IV-10) and (IV-11) are exactly those obtained by Reissner<sup>5</sup>. From (IV-8) and (IV-9), it is clear that (IV-10) and (IV-11) are valid if

$$\frac{\alpha}{\mu} \left| w_n \left[ 1 + \frac{(n+1)\nu}{2\mu} \right] - \frac{a\beta_n}{2\mu} \left[ 1 + \frac{(n+1)(1+\nu) - (2n+1)}{2\mu} \right] \right| \ll |u_n + v_n| \quad (IV-12)$$

and

$$\frac{|u_n + v_n|}{\mu} \ll |u_n - v_n| \quad (IV-13)$$

### C. Interior Membrane and Inextensional Bending Stresses

To examine the relative importance of the membrane and inextensional bending stresses in the interior of the shell, we consider the ratio

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{B_n}{hA_n}\right) \quad (IV-14)$$

From (IV-8) and (IV-9), we have

$$\begin{aligned} \frac{B_n}{hA_n} = & -\frac{n(n+1)(1-\nu^2)a^2}{4\mu^4 Rh} \left( (u_n + v_n) \left[1 + O\left(\frac{1}{\mu}\right)\right] - \frac{\alpha(1+\nu)}{\mu} \left\{ \left(w_n - \frac{a\beta_n}{2\mu}\right) \right. \right. \\ & \times \left. \left. \left[1 + \frac{(n+1)\nu}{2\mu} + O\left(\frac{1}{\mu^2}\right)\right] - \frac{na\beta_n}{(2\mu)^2} \left[1 + O\left(\frac{1}{\mu}\right)\right] \right\} \right) / \left( (u_n - v_n) \left[1 + O\left(\frac{1}{\mu^2}\right)\right] \right. \\ & + \frac{(n+1)(1+\nu)}{2\mu} (u_n + v_n) \left[1 + O\left(\frac{1}{\mu}\right)\right] - \frac{\alpha(1+\nu)}{\mu} \left\{ \left(w_n - \frac{a\beta_n}{2\mu}\right) \left[1 + \frac{(n+1)\nu + (n-1)}{2\mu} \right. \right. \\ & \left. \left. + O\left(\frac{1}{\mu^2}\right)\right] - \frac{na\beta_n}{2\mu} \left[1 + O\left(\frac{1}{\mu}\right)\right] \right\} \right) \quad (IV-15) \end{aligned}$$

For a given set of  $u_n$ ,  $v_n$ ,  $w_n$ , and  $\beta_n$ , the relative order of magnitude of  $\sigma_B$  and  $\sigma_D$  can be determined by way of (IV-14) and (IV-15). In what follows, we shall restrict ourselves to isotropic and homogeneous shells so that

$$\frac{a^2}{Rh} = \frac{\mu^2}{\sqrt{3(1-\nu^2)}}$$

Consider the class of problems for which  $w_n = \beta_n = 0$  so that (IV-15) becomes

$$\frac{B_n}{hA_n} = O\left\{ \frac{1}{\mu^2} \left[ \frac{(u_n + v_n) \left[1 + O\left(\frac{1}{\mu}\right)\right]}{(u_n - v_n) \left[1 + O\left(\frac{1}{\mu^2}\right)\right] + \frac{(n+1)(1+\nu)}{2\mu} (u_n + v_n) \left[1 + O\left(\frac{1}{\mu}\right)\right]} \right] \right\} \quad (IV-16)$$

(1) If  $|u_n + v_n|/\mu \ll |u_n - v_n|$ , we have

$$\frac{B_n}{hA_n} = O\left[ \frac{1}{\mu^2} \left( \frac{u_n + v_n}{u_n - v_n} \right) \right] \quad (IV-17)$$

Therewith,

$$\frac{\sigma_B}{\sigma_D} \ll \frac{1}{\mu} \quad (IV-18)$$

The interior of the shell is therefore in a membrane stress state.

(2) If  $(u_n + v_n)/\mu$  and  $(u_n - v_n)$  are of the same order of magnitude, unless

$$|(u_n - v_n) + \frac{(n+1)(1+\nu)}{2\mu} (u_n + v_n)| \ll |u_n + v_n| \quad , \quad (\text{IV-19})$$

we have

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{1}{\mu}\right) \quad . \quad (\text{IV-20})$$

For the exception (if (IV-19) holds), we must consider the relevant third-order terms to determine whether the interior stress state is a mixed state or an inextensional bending state.

(3) If  $|u_n - v_n| \ll |u_n + v_n|/\mu$ ,

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{1}{\mu}\right) \quad (\text{IV-21})$$

and again we have a membrane interior stress state.

The above results should be compared with those given in Ref. 5.

Consider the class of problems for which  $u_n = v_n = 0$ , so that

$$\frac{B_n}{hA_n} = O\left\{\frac{1}{\mu^2} \left[ \frac{\left(w_n - \frac{a\beta_n}{2\mu}\right) \left[1 + \frac{(n+1)\nu}{2\mu} + O\left(\frac{1}{\mu^2}\right)\right] - \frac{na\beta_n}{(2\mu)^2} [1 + O\left(\frac{1}{\mu}\right)]}{\left(w_n - \frac{a\beta_n}{2\mu}\right) \left[1 + \frac{2n + (n+1)\nu}{2\mu} + O\left(\frac{1}{\mu^2}\right)\right] - \frac{na\beta_n}{(2\mu)^2} [1 + O\left(\frac{1}{\mu}\right)]} \right] \right\} \quad . \quad (\text{IV-22})$$

It is not difficult to see [with the help of (IV-22)] that the interior of the shell for this class of problems is almost always in a membrane stress state. The only exception occurs when  $w_n - (a\beta_n/2\mu)$  and  $a\beta_n/2\mu^2$  are of the same order of magnitude while

$$\left(w_n - \frac{a\beta_n}{2\mu}\right) - \frac{na\beta_n}{(2\mu)^2} \ll \frac{a\beta_n}{(2\mu)^2} \quad . \quad (\text{IV-23})$$

For this exception, it may be necessary to ascertain the relevant third-order terms in the various asymptotic expansion in order to determine the interior stress state.

## V. PROBLEM OF PRESCRIBED TANGENTIAL EDGE DISPLACEMENTS AND TRANSVERSE EDGE LOAD AND MOMENT

### A. Prescribed Conditions at Edge

Consider now a shell without surface loads but with the following constraints at the edge  $\rho = 1$ :

$$\begin{aligned} v &= v_n \sin n\theta \quad , \quad u = u_n \cos n\theta \quad , \\ M_r &= M_n \cos n\theta \quad , \quad R_r = R_n \cos n\theta \quad , \end{aligned} \quad (\text{V-1})$$

where  $n \geq 2$ . Substituting into (V-1) the appropriate expressions for  $u$ ,  $v$ ,  $M_r$ , and  $R_r$  from (II-18) evaluated at  $\rho = 1$ , we get

$$na(1+\nu) AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} - \frac{n(1+\nu)\sqrt{A}}{a} [C_n \text{bei}_n(\lambda) - D_n \text{ber}_n(\lambda)] = v_n$$

$$\begin{aligned}
-n a(1+\nu) A A_n + \frac{a^3 B_n}{R D(n+1)(1-\nu)} + \frac{(1+\nu) \lambda \sqrt{A}}{a} [C_n \text{bei}'_n(\lambda) - D_n \text{ber}'_n(\lambda)] &= u_n \\
-n(n-1) B_n + \frac{1}{R \sqrt{A}} [C_n g_{rc}(\lambda) - D_n g_{rd}(\lambda)] &= M_n \\
n^2(n-1) B_n + \frac{\lambda}{R \sqrt{A}} [C_n f_{nc}(\lambda) - D_n f_{nd}(\lambda)] &= a R_n \quad .
\end{aligned} \tag{V-2}$$

Solving (V-2) for  $A_n$  and  $B_n$ , we get

$$A_n = \frac{1}{2n\alpha a} \{nM_n Z_1 + aR_n Z_2 + \frac{\alpha}{A(1+\nu)} [v_n Z_3 - u_n Z_4]\} \tag{V-3}$$

$$B_n = \frac{(n+1)(1-\nu^2)}{2\lambda^4} [nM_n Z_5 - aR_n Z_6 + \frac{\alpha}{A(1+\nu)} (u_n + v_n) Z_7] \tag{V-4}$$

where

$$\begin{aligned}
Z_1 &= \frac{1}{\Delta_3} \left\{ \left[ 1 - \frac{n^2(n^2-1)(1-\nu^2)}{\lambda^4} \right] + \frac{n(1-\nu)}{\lambda} \left( \alpha_2 + \frac{n-1}{\lambda} \alpha_3 - \frac{n}{\lambda^2} \alpha_4 \right) \right\} \\
Z_2 &= \frac{1}{\Delta_3} \left\{ \left[ 1 - \frac{n^2(n^2-1)(1-\nu^2)}{\lambda^4} \right] - \frac{1-\nu}{\lambda} \left[ \alpha_2 - \frac{n(n-1)}{\lambda} \alpha_3 - \frac{n^3}{\lambda^2} \alpha_4 \right] \right\} \\
Z_3 &= \frac{1}{\Delta_3} \left( \left[ 1 - \frac{2n^2(n^2-1)(1-\nu)}{\lambda^4} \right] - \frac{1-\nu}{\lambda} \left\{ \alpha_2 \left[ 1 - \frac{n^2(n+1)(n^2-1)(1-\nu^2)}{\lambda^4} \right] \right. \right. \\
&\quad \left. \left. - \frac{2n^2}{\lambda} \alpha_3 \left[ 1 - \frac{n(n+1)(n^2-1)(1-\nu)^2}{2\lambda^4} \right] + \frac{n^2}{\lambda^2} \alpha_4 \right\} \right) \\
Z_4 &= \frac{1}{\Delta_3} \left( \left[ 1 - \frac{2n^2(n^2-1)(1-\nu)}{\lambda^4} \right] - \frac{1-\nu}{\lambda} \left\{ \alpha_2 - \frac{2n^2}{\lambda} \alpha_3 \left[ 1 - \frac{n(n+1)(n^2-1)(1-\nu^2)}{2\lambda^4} \right] \right. \right. \\
&\quad \left. \left. + \frac{n^2}{\lambda^2} \alpha_4 \left[ 1 - \frac{n^2(n+1)(n^2-1)(1-\nu^2)}{\lambda^4} \right] \right\} \right) \\
Z_5 &= \frac{1}{\Delta_3} \left\{ 1 - \frac{n(1-\nu)}{\lambda} \left[ \alpha_2 - \frac{(n+1)}{\lambda} \alpha_3 + \frac{n}{\lambda} \alpha_4 \right] \right\} \\
Z_6 &= \frac{1}{\Delta_3} \left\{ 1 - \frac{1-\nu}{\lambda} \left[ \alpha_2 - \frac{n(n-1)}{\lambda} \alpha_3 - \frac{n^3}{\lambda} \alpha_4 \right] \right\} \\
Z_7 &= \frac{1}{\Delta_3} \left\{ \left[ 1 - \frac{n^2(n^2-1)(1-\nu^2)}{\lambda^4} \right] - \frac{1-\nu}{\lambda} \left( \alpha_2 - \frac{2n^2}{\lambda} \alpha_3 + \frac{n^2}{\lambda^2} \alpha_4 \right) \right\}
\end{aligned}$$

with

$$\begin{aligned}
\Delta_3 &= \left[ 1 - \frac{2n^2(n^2-1)(1-\nu)}{\lambda^4} \right] - \frac{1-\nu}{\lambda} \left\{ \alpha_2 \left[ 1 - \frac{n^2(n^2-1)(n+1)(1-\nu^2)}{2\lambda^4} \right] \right. \\
&\quad \left. - \frac{2n^2}{\lambda} \alpha_3 \left[ 1 - \frac{n(n^2-1)(n+1)(1-\nu^2)}{2\lambda^4} \right] + \frac{n^2}{\lambda^2} \alpha_4 \left[ 1 - \frac{n^2(n^2-1)(n+1)(1-\nu^2)}{2\lambda^4} \right] \right\}
\end{aligned} \tag{V-5}$$

## B. Asymptotic Interior Solution

For  $\mu \gg n > 1$ , the following asymptotic expressions for the  $Z_i$ 's may be obtained with the help of (III-12) and (III-13):

$$\begin{aligned}
Z_1 &\sim 1 + \frac{2(n+1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \\
Z_2 &\sim 1 - \frac{2n(n+1)(1-\nu)}{(2\mu)^2} + O\left(\frac{1}{\mu^3}\right) \\
Z_3 &\sim 1 + \frac{4n^2(n+1)^2(n-1)(1-\nu)^2(1+\nu)}{(2\mu)^5} + O\left(\frac{1}{\mu^6}\right) \\
Z_4 &\sim 1 - \frac{4n^2(n+1)^2(n-1)(1-\nu)^2(1+\nu)}{(2\mu)^5} + O\left(\frac{1}{\mu^6}\right) \\
Z_5 &\sim 1 - \frac{2(n-1)(1-\nu)}{(2\mu)} + O\left(\frac{1}{\mu^2}\right) \\
Z_6 &\sim 1 - \frac{2n(n+1)(1-\nu)}{(2\mu)^2} + O\left(\frac{1}{\mu^2}\right) \\
Z_7 &\sim 1 + \frac{4n^2(n^2-1)(1-\nu)}{(2\mu)^4} + O\left(\frac{1}{\mu^5}\right) \quad . \quad (V-6)
\end{aligned}$$

At least two terms have been retained from the expansions, regardless of the order of the second term. It was shown earlier that these second terms may become significant under certain circumstances. Correspondingly we have

$$\begin{aligned}
A_n &\sim \frac{1}{2n\alpha a} \left\{ aR_n \left[ 1 - \frac{2n(n+1)(1-\nu)}{(2\mu)^2} \right] + nM_n \left[ 1 + \frac{2(n+1)(1-\nu)}{2\mu} \right] \right. \\
&\quad - \frac{\alpha u_n}{A(1+\nu)} \left[ 1 - \frac{4n^2(n+1)^2(n-1)(1-\nu)^2(1+\nu)}{(2\mu)^5} \right] \\
&\quad \left. + \frac{\alpha v_n}{A(1+\nu)} \left[ 1 + \frac{4n^2(n+1)^2(n-1)(1-\nu)^2(1+\nu)}{(2\mu)^5} \right] \right\} \quad , \quad (V-7)
\end{aligned}$$

$$\begin{aligned}
B_n &\sim \frac{-2(n+1)(1-\nu^2)}{(2\mu)^4} \left\{ aR_n \left[ 1 - \frac{2n(n+1)(1-\nu)}{(2\mu)^2} \right] \right. \\
&\quad \left. - nM_n \left[ 1 - \frac{2(n-1)(1-\nu)}{2\mu} \right] - \frac{\alpha(u_n + v_n)}{A(1+\nu)} \left[ 1 + \frac{4n^2(n^2-1)(1-\nu^2)}{(2\mu)^4} \right] \right\} \quad . \quad (V-8)
\end{aligned}$$

We shall also include for later reference the expressions for  $A_n$  and  $B_n$ , retaining only the leading term from the expansions for the  $Z_i$ 's,

$$A_n \sim \frac{1}{2n\alpha a} \left[ (aR_n + nM_n) - \frac{\alpha(u_n - v_n)}{A(1+\nu)} \right] \quad (V-9)$$

$$B_n \sim \frac{-2(n+1)(1-\nu^2)}{(2\mu)^4} \left[ (aR_n - nM_n) - \frac{\alpha(u_n + v_n)}{A(1+\nu)} \right] \quad . \quad (V-10)$$

Clearly, (V-9) and (V-10) are valid only if

$$\begin{aligned} & \frac{n(n+1)(1-\nu)}{2\mu} \left| M_n - \frac{aR_n}{\mu} + \frac{2n(n^2-1)(1+\nu)}{(2\mu)^4} \frac{\alpha}{A} (u_n + v_n) \right| \\ & \ll \left| (aR_n + nM_n) - \frac{\alpha(u_n - v_n)}{A(1+\nu)} \right| \end{aligned} \quad (V-11)$$

and

$$\begin{aligned} & \frac{n(1-\nu)}{2\mu} \left| (n-1) M_n - \frac{(n+1)aR_n}{\mu} - \frac{2n(n^2-1)}{(2\mu)^3} \frac{\alpha}{A} (u_n + v_n) \right| \\ & \ll \left| (aR_n - nM_n) - \frac{\alpha}{A} \frac{(u_n + v_n)}{(1+\nu)} \right| \end{aligned} \quad (V-12)$$

### C. Direct Derivation of Interior Solution

Although the particular problem under consideration has not been solved explicitly, the procedure described in Ref. 4 is clearly applicable. In this section, an approximate interior solution will be obtained by the aforementioned procedure to see its relation to the exact solution given in Sec. V-A.

The meridional and circumferential displacement components  $u$  and  $v$  can be expressed in terms of the stress function  $F$  and the normal displacement  $w$  by way of the strain-deformation relations (II-4) as

$$\begin{aligned} u &= \frac{1}{R} \int \psi dr - A(1+\nu) F_{,r} = u^i - (1+\nu) RAD\xi_{,r} \\ v &= -\frac{r}{R} \left[ \int \left( \int \psi dr \right)_{,\Theta} \frac{dr}{r^2} \right] - \frac{A(1+\nu) F_{,\Theta}}{r} = v^i - \frac{(1+\nu) RAD\xi_{,\Theta}}{r} \end{aligned}$$

where

$$\xi = \nabla^2 \chi$$

The quantities  $u^i$  and  $v^i$ , which are the portion of  $u$  and  $v$  corresponding to the interior state, are given by

$$\begin{aligned} u^i &= \frac{1}{R} \int \psi dr - A(1+\nu) \varphi_{,r} \\ v^i &= -\frac{r}{R} \int \left( \int \psi dr \right)_{,\Theta} \frac{dr}{r^2} - \frac{A(1+\nu) \varphi_{,\Theta}}{r} \end{aligned}$$

Corresponding expressions for  $M_r$  and  $R_r$  in terms of  $\varphi$ ,  $\psi$ ,  $\chi$  and  $\xi$  are

$$\begin{aligned} M_r &= -D \left[ w_{,rr} + \nu \left( \frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\Theta\Theta} \right) \right] \\ &= M_r^i - D \left[ \xi - (1-\nu) \left( \frac{1}{r} \chi_{,r} + \frac{1}{r^2} \chi_{,\Theta\Theta} \right) \right] \\ R_r &= -D \left[ (\nabla^2 w)_{,r} + \frac{1-\nu}{r} \left( \frac{1}{r} w_{,\Theta} \right)_{,r} \right] \\ &= R_r^i - D \left[ \xi_{,r} + \frac{1-\nu}{r} \left( \frac{1}{r} \chi_{,\Theta} \right)_{,r} \right] \end{aligned} \quad (V-13)$$

where

$$M_r^i = -D \left[ \psi_{,rr} + \nu \left( \frac{1}{r} \psi_{,r} + \frac{1}{r^2} \psi_{,\theta\theta} \right) \right]$$

and

$$R_r^i = -D \left[ \frac{1-\nu}{r} \left( \frac{1}{r} \psi_{,\theta} \right)_{,r} \right] \quad (V-14)$$

are the portion of  $M_r$  and  $R_r$  corresponding to the interior state. We now approximate  $M_r$  and  $R_r$  by discarding all terms involving  $\chi$  explicitly and we are left with

$$M_r \approx M_r^i - D\xi \quad , \quad R_r \approx R_r^i - D\xi_{,r} \quad . \quad (V-15)$$

(For a justification of this approximation, the reader is referred to Refs. 4 and 9.) At the edge  $\rho = 1$ , we have then

$$\begin{aligned} u^i - (1+\nu) \text{RAD} \xi_{,r} &= u_n \cos n\theta \quad , \quad v^i - \frac{(1+\nu) \text{RAD}}{a} \xi_{,\theta} = v_n \sin n\theta \quad , \\ M_r^i - D\xi &= M_n \cos n\theta \quad , \quad R_r^i - D\xi_{,r} = R_n \cos n\theta \quad . \end{aligned} \quad (V-16)$$

Now  $\xi$  can be eliminated from these relations and we are left with

$$RA(1+\nu) R_r^i - u^i = [RA(1+\nu) R_n - u_n] \cos n\theta$$

and

$$\frac{RA(1+\nu)}{a} (M_r^i)_{,\theta} - v^i = \left[ \frac{RA(1+\nu) n M_n}{a} - v_n \right] \sin n\theta \quad . \quad (V-17)$$

Substituting into the above two relations the interior portion of the solution listed in (II-18) and upon solving for  $A_n$  and  $B_n$ , we get

$$A_n = \frac{1}{2n\alpha a} \left[ (aR_n + nM_n) - \frac{\alpha(u_n - v_n)}{A(1+\nu)} \right] \quad (V-18)$$

$$B_n = \frac{-2(n+1)(1-\nu^2)}{(2\mu)^4 \left[ 1 - \frac{n^2(n^2-1)(1-\nu^2)}{(2\mu)^2} \right]} \left[ (aR_n - nM_n) - \frac{\alpha(u_n + v_n)}{A(1+\nu)} \right] \quad . \quad (V-19)$$

Suppressing terms of the order  $1/\mu$  in the presence of unity, (V-18) and (V-19) are exactly the same as (V-9) and (V-10).

#### D. Interior Membrane and Inextensional Bending Stresses

Although a detailed discussion similar to that given in Sec. IV-C can be carried out for homogeneous and isotropic shells, we shall limit ourselves in what follows to two special cases for the purpose of comparison with the earlier problems.

For a shell which is free from transverse edge load and moment so that  $R_n = M_n = 0$ , we have, from (V-7) and (V-8), for  $(\mu \gg n > 1)$ ,

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{B_n}{hA_n}\right) = O\left\{ \mu^2 \left[ \frac{(u_n - v_n)}{(u_n + v_n)} + \frac{K}{(2\mu)^5} \right]^{-1} \right\} \quad , \quad (V-20)$$

where

$$K = n^2(n+1)^2(n-1)(1-\nu)^2(1+\nu) \quad .$$

Thus, similar to shells with only prescribed tangential edge loads (Sec. III-D), the interior of the shell may be in a membrane, a mixed or an inextensional bending stress state depending on the relative magnitude of the quantities  $(u_n + v_n)$  and  $(u_n - v_n)$ .

On the other hand, if the shell is constrained in the tangential directions so that  $u_n = v_n = 0$ , we have (for  $\mu \gg n > 1$ )

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{B_n}{hA_n}\right) = O\left\{\frac{1}{\mu^2} \frac{(aR_n - nM_n) + \frac{n(1-\nu)}{\mu} [(n-1)M_n - \frac{n+1}{2\mu} aR_n]}{(aR_n + nM_n) + \frac{n(n+1)(1-\nu)}{\mu} \left(M_n - \frac{aR_n}{2\mu}\right)}\right\} \quad . \quad (V-21)$$

Thus, similar to shells with prescribed transverse displacement and meridional change in slope at the edge (along with vanishing tangential edge displacements), the interior of the shell is almost always in a membrane state (Sec. IV-C). The only exception occurs when

$$aR_n + nM_n = O\left(\frac{M_n}{\mu}\right) \quad (V-22)$$

and

$$\left| (aR_n + nM_n) + \frac{n(n+1)(1-\nu)}{\mu} \left(M_n - \frac{aR_n}{2\mu}\right) \right| \ll |aR_n - nM_n| \quad , \quad (V-23)$$

in which case the relevant third-order terms in  $A_n$  may have to be considered in order to determine whether the interior is in a mixed or an inextensional bending stress state.

Note that these results are in agreement with our earlier observation (Sec. III-D). They demonstrate once again that while self-equilibrating tangential edge loads alone may lead to any one of the three possible interior stress states, it takes a rather special combination of self-equilibrating transverse edge load and edge moment to induce an interior stress state other than an inextensional bending state.

## VI. PROBLEM OF PRESCRIBED TANGENTIAL EDGE LOADS AND TRANSVERSE EDGE MOMENT AND DEFLECTION

### A. Prescribed Conditions at Edge

In this section, we consider a shell with the following edge conditions at  $\rho = 1$ :

$$\begin{aligned} N_r &= N_n \cos n\theta \quad , & N_{r\theta} &= S_n \sin n\theta \quad , \\ M_r &= M_n \cos n\theta \quad , & w &= w_n \cos n\theta \quad , \end{aligned} \quad (VI-1)$$

where  $n \geq 2$ . Note that (VI-1) differs from (III-1) only in the last condition; the prescription of the transverse force  $R_r$  in (III-1) is now replaced by that of the transverse deflections  $w$ . For fixed values of  $N_n$ ,  $M_n$ , and  $S_n$ , the results of this section and those of Sec. III are identical, if the  $R_n$  is taken to be exactly that amount of transverse force needed to produce the transverse displacement  $w_n$ . On the other hand, the explicit asymptotic interior solution for (VI-1) will make some rather interesting conclusions easily accessible.

Observing (II-18), (VI-1) becomes

$$\begin{aligned}
\frac{a^2}{D(1-\nu)} B_n + \frac{1}{\sqrt{D}} [C_n \text{ber}_n(\lambda) + D_n \text{bei}_n(\lambda)] &= w_n \\
-n(n-1) B_n + \frac{1}{R\sqrt{A}} [C_n g_{rc}(\lambda) - D_n g_{rd}(\lambda)] &= M_n \\
n(n-1) A_n + \frac{\lambda}{a^2\sqrt{A}} [C_n f_{rc}(\lambda) - D_n f_{rd}(\lambda)] &= -N_n \\
n(n-1) A_n - \frac{n\lambda}{a^2\sqrt{A}} [C_n f_{sc}(\lambda) - D_n f_{sd}(\lambda)] &= S_n \quad . \quad (VI-2)
\end{aligned}$$

Solving (VI-2) for  $A_n$  and  $B_n$ , we get

$$\begin{aligned}
A_n = \frac{-1}{\alpha a} \left\{ \left[ M_n + \frac{n(n-1)(1-\nu)D}{a^2} w_n \right] T_1 \right. \\
\left. + \frac{\alpha a}{n(n^2-1)} [(nN_n) T_2 - S_n T_3] \right\} \quad (VI-3)
\end{aligned}$$

$$B_n = \frac{1-\nu}{n+1} \left[ \frac{(n+1)Dw_n}{a^2} T_4 + M_n T_5 - \alpha a(S_n + N_n) T_6 \right] \quad (VI-4)$$

where

$$\begin{aligned}
T_1 &= \frac{1}{\Delta_4} \\
T_2 &= \frac{1}{\Delta_4} \left[ 1 - \frac{1-\nu}{\lambda} \left( \alpha_2 - \frac{n+1}{\lambda} \alpha_3 + \frac{n}{\lambda^2} \alpha_4 \right) \right] \\
T_3 &= \frac{1}{\Delta_4} \left\{ 1 - \frac{1-\nu}{\lambda} \left[ \alpha_2 - \frac{n(n+1)}{\lambda} \alpha_3 + \frac{n^3}{\lambda^2} \alpha_4 \right] \right\} \\
T_4 &= \frac{1}{\Delta_4} \left\{ 1 - \frac{1-\nu}{\lambda} \left[ \alpha_2 - \frac{n(n+1)}{\lambda} \alpha_3 + \frac{n^3}{\lambda^2} \alpha_4 \right] \right\} \\
T_5 &= \frac{1}{\lambda^2 \Delta_4} (\alpha_3 - \frac{n}{\lambda} \alpha_4) \\
T_6 &= \frac{1}{\lambda^3 \Delta_4} \left[ \alpha_4 - \frac{(1-\nu)}{\lambda} \right]
\end{aligned}$$

with

$$\Delta_4 = 1 - \frac{1-\nu}{\lambda} \left( \alpha_2 - \frac{2n}{\lambda} \alpha_3 + \frac{n^2}{\lambda^2} \alpha_4 \right) \quad . \quad (VI-5)$$

## B. Asymptotic Interior Solution

For  $\mu \gg n > 1$ , the following asymptotic expressions for the  $T_i$ 's may be obtained with the help of (III-12) and (III-13):

$$\begin{aligned}
T_1 &\sim 1 + \frac{2(1-\nu)}{2\mu} + O\left(\frac{1}{\mu}\right) \\
T_2 &\sim 1 - \frac{2(n-1)(1-\nu)}{(2\mu)^2} + O\left(\frac{1}{\mu^3}\right) \\
T_3 &\sim 1 + \frac{2n(n-1)(1-\nu)}{(2\mu)^2} + O\left(\frac{1}{\mu^3}\right) \\
T_4 &\sim 1 + \frac{2n(n-1)(1-\nu)}{(2\mu)^2} + O\left(\frac{1}{\mu^3}\right) \\
T_5 &\sim \frac{2}{(2\mu)^2} \left[ 1 - \frac{1}{2\mu} [(2n-1) + 2\nu] + O\left(\frac{1}{\mu^2}\right) \right] \\
T_6 &\sim \frac{4}{(2\mu)^3} \left[ 1 + \frac{1-\nu}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \quad . \quad (VI-6)
\end{aligned}$$

Correspondingly, we have

$$\begin{aligned}
A_n &\sim -\frac{1}{\alpha a} \left( \left[ M_n + \frac{n(n-1)(1-\nu)D}{a^2} w_n \right] \left[ 1 + \frac{2(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] + \frac{\alpha a}{n(n^2-1)} \right. \\
&\quad \left. \times \left[ nN_n \left[ 1 - \frac{2(n-1)(1-\nu)}{(2\mu)^2} + O\left(\frac{1}{\mu^3}\right) \right] - S_n \left[ 1 + \frac{2n(n-1)(1-\nu)}{(2\mu)^2} + O\left(\frac{1}{\mu^3}\right) \right] \right] \right) \quad (VI-7)
\end{aligned}$$

and

$$\begin{aligned}
B_n &\sim (1-\nu) \left\{ \left[ 1 + \frac{2n(n-1)(1-\nu)}{(2\mu)^2} + O\left(\frac{1}{\mu^2}\right) \right] \left( \frac{Dw_n}{a} \right) + \left[ 1 - \frac{2n-(1-2\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \right. \\
&\quad \left. \times \left[ \frac{2M_n}{(2\mu)^2} - \frac{4\alpha a}{(n+1)(2\mu)^3} \left[ 1 + \frac{1-\nu}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] (S_n + N_n) \right] \right\} \quad . \quad (VI-8)
\end{aligned}$$

If only the leading term of the expansions is retained, (VI-7) and (VI-8) become

$$A_n \sim -\frac{1}{\alpha a} \left\{ \left[ \frac{n(n-1)(1-\nu)D}{a^2} w_n + M_n \right] + \frac{\alpha a}{n(n^2-1)} (nN_n - S_n) \right\} \quad (VI-9)$$

and

$$B_n \sim (1-\nu) \left[ \frac{D}{a} w_n + \frac{2(n+1)}{(2\mu)^2} M_n - \frac{4\alpha a}{(n+1)(2\mu)^3} (S_n + N_n) \right] \quad . \quad (VI-10)$$

respectively. Clearly (VI-9) and (VI-10) are valid as long as

$$\begin{aligned}
\frac{1}{\mu} \left| M_n + \frac{n(n-1)(1-\nu)D}{a^2} w_n - \frac{\alpha a}{n+1} \frac{(S_n + N_n)}{2\mu} \right| &\ll \left| M_n + \frac{n(n-1)(1-\nu)D}{a^2} w_n \right. \\
&\quad \left. + \frac{\alpha a}{n(n^2-1)} (nN_n - S_n) \right| \quad (VI-11)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\mu^2} \left| \frac{n(n-1)(1-\nu)D}{a^2} w_n - \frac{2n-(1-2\nu)}{2\mu} M_n - \frac{\alpha a(1-\nu)}{2\mu^2(n+1)} (S_n + N_n) \right| \\
\ll \left| \frac{D}{a} w_n + \frac{2(n+1)}{(2\mu)^2} M_n - \frac{4\alpha a}{(n+1)(2\mu)^3} (S_n + N_n) \right| \quad . \quad (VI-12)
\end{aligned}$$

### C. Interior Membrane and Inextensional Bending Stresses

From (VI-7) and (VI-8), we have

$$\begin{aligned}
\frac{B_n}{hA_n} = & -\frac{\alpha a(1-\nu)}{h} \left\{ \left( \frac{Dw_n}{a^2} + \frac{M_n}{2\mu^2} \right) \left[ 1 + O\left(\frac{1}{\mu^2}\right) \right] + \frac{n(n-1)(1-\nu)}{2\mu^2} \frac{Dw_n}{a^2} - \frac{2n-(1-2\nu)}{4\mu^3} \right. \\
& \times M_n - \frac{\alpha a}{(n+1)2\mu^3} (S_n + N_n) \left. \left[ 1 + O\left(\frac{1}{\mu}\right) \right] \right\} \Bigg/ \left\{ \left[ \frac{n(n-1)(1-\nu)D}{a^2} w_n + M_n \right] \right. \\
& \times \left[ 1 + O\left(\frac{1}{\mu}\right) \right] + \frac{\alpha a}{n(n^2-1)} [nN_n - S_n] \left. \left[ 1 + O\left(\frac{1}{\mu^3}\right) \right] - \frac{\alpha a(1-\nu)}{2\mu^2(n+1)} \right. \\
& \left. \times (N_n + S_n) \left[ 1 + O\left(\frac{1}{\mu}\right) \right] \right\} . \tag{VI-13}
\end{aligned}$$

A rather unique feature of the present problem now reveals itself. In all the problems considered earlier, if we set all but one of the prescribed quantities equal to zero, the interior stress state remains the same for each problem regardless of which quantity is nonzero. This is not so for the present problem.

If  $M_n = N_n = S_n = 0$ ,

$$\frac{B_n}{hA_n} = -\frac{\mu^2}{n(n-1)\sqrt{3(1-\nu^2)}} \tag{VI-14}$$

whence  $\sigma_B \gg \sigma_D$  in the interior of the shell.

If  $w_n = S_n = N_n = 0$ ,

$$\frac{B_n}{hA_n} = -\frac{1-\nu}{\sqrt{3(1-\nu^2)}} \left[ 1 + O\left(\frac{1}{\mu}\right) \right] . \tag{VI-15}$$

$\sigma_B$  and  $\sigma_D$  are of the same order of magnitude in this case.

If  $w_n = M_n = S_n = 0$ ,

$$\frac{B_n}{hA_n} = \frac{(n-1)(1-\nu)}{\mu\sqrt{3(1-\nu^2)}} \left[ 1 + O\left(\frac{1}{\mu}\right) \right] . \tag{VI-16}$$

Therewith,

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{1}{\mu}\right) \tag{VI-17}$$

and we have a membrane interior stress state.

In arriving at (VI-14), (VI-15), and (VI-16), we have assumed that the shell is homogeneous and isotropic so that

$$\frac{\alpha a}{h} = \frac{\mu^2}{\sqrt{3(1-\nu^2)}} . \tag{VI-18}$$

Under the same assumption, we now consider the class of problems for which  $w_n = M_n = 0$ . For these problems, (VI-13) reduces to

$$\frac{B_n}{hA_n} = \frac{n(n-1)(1-\nu)}{2\mu\sqrt{3(1-\nu^2)}} \times \left\{ \frac{(S_n + N_n) \left[1 + O\left(\frac{1}{\mu}\right)\right]}{\left(nN_n - S_n\right) \left[1 + O\left(\frac{1}{\mu^3}\right)\right] - \frac{n(n-1)(1-\nu)}{2\mu^2} (S_n + N_n) \left[1 + O\left(\frac{1}{\mu}\right)\right]} \right\} \quad (\text{VI-19})$$

All three stress states are possible in the interior of the shell, depending on the magnitude of  $(S_n + N_n)/(nN_n - S_n)$ . Equation (VI-19) should be compared with the corresponding result given in Ref. 4.

## VII. SHELLS WITH POLAR HARMONIC AXIAL SURFACE LOADS

### A. Solution to Differential Equations

Let the surface load intensity vector be axial with magnitude

$$P_z(r, \theta) = P_0 \rho^n \cos n\theta \quad (n \geq 1)$$

Within the context of shallow shell theory, an axial and a normal surface load are completely equivalent.<sup>5</sup> Thus we may write

$$\begin{aligned} P_n &= P_z = P_0 \rho^n \cos n\theta \quad (n \geq 1) \\ P_r &\equiv P_\theta \equiv 0 \end{aligned} \quad (\text{VII-1})$$

The particular solutions to (II-1) with  $P_n$  as given in (VII-1) are

$$\begin{aligned} F_p &= -\frac{P_0 a^2 R}{4(n+1)} \rho^{n+2} \cos n\theta \\ w_p &= P_0 A R^2 \rho^n \cos n\theta \end{aligned} \quad (\text{VII-2})$$

Although the particular solution for  $w$  has the same form as the inextensional bending component of the homogeneous solution, and therefore may be omitted from further consideration, it will be kept throughout the subsequent development to facilitate later comparisons with existing results.

The corresponding particular solutions for the resultants, couples, tangential displacement components, and meridional slope change are:

$$\begin{aligned} N_{rp} &= \frac{n-2}{4} P_0 R \rho^n \cos n\theta \quad , \quad N_{\theta p} = \frac{n+2}{4} P_0 R \rho^n \cos n\theta \quad , \\ N_{r\theta p} &= -\frac{n}{4} P_0 R \rho^n \sin n\theta \quad , \quad Q_{rp} \equiv Q_{\theta p} \equiv 0 \quad , \\ M_{rp} &= -M_{\theta p} = -n(n-1)(1-\nu) \frac{DAR^2}{a^2} P_0 \rho^{n-2} \cos n\theta \quad , \end{aligned}$$

$$\begin{aligned}
M_{r\Theta p} &= n(n-1)(1-\nu) \frac{DAR^2}{a^2} P_0 \rho^{n-2} \sin n\Theta \quad , \\
R_{rp} &= n^2(n-1)(1-\nu) \frac{DAR^2}{a^3} P_0 \rho^{n-3} \cos n\Theta \quad , \\
v_p &= -\frac{n(1+\nu)}{4(n+1)} P_0 aAR \rho^{n+1} \sin n\Theta \quad , \\
u_p &= \frac{(n+2)(1+\nu)}{4(n+1)} P_0 aAR \rho^{n+1} \cos n\Theta \quad , \\
\frac{\partial w_p}{\partial r} &= \frac{n}{a} P_0 AR^2 \rho^{n-1} \cos n\Theta \quad . \tag{VII-3}
\end{aligned}$$

The complete solutions to the problem can now be obtained by summing these particular solutions and the corresponding homogeneous solutions given in (II-18). The four constants of integration which appear in the resulting expressions are to be determined by the constraints at the edge  $\rho = 1$ . For  $n = 1$ , our results are just those obtained by Reissner.<sup>5\*</sup> Observe that in this case, the particular solutions for the transverse shears and the bending moments vanish identically. Moreover, the interior solutions for these quantities and for the in-plane stress resultants also vanish identically.

In the next few sections, the nature of the interior solution for shells with various types of support will be considered. In particular, we shall show that, to the first approximation, the interior stresses of a shell with a clamped edge and one with a simply supported edge are the same, and that the results from a membrane and inextensional bending analysis are equivalent to this first approximation of the interior solution. For  $n = 1$ , the membrane solution turns out to be the exact solution to (II-1) for a shell fixed tangentially but free otherwise. The interior of the shell in all cases is dominated by the membrane stresses. On the other hand, for  $n \geq 2$ , the interior of a shell with free edge is primarily in a state of inextensional bending.

### B. Clamped Edge

Consider first a shell with a clamped edge so that

$$u = v = w = \frac{\partial w}{\partial r} = 0 \quad (\text{at } \rho = 1) \quad . \tag{VII-4}$$

Substituting into (VII-4) the appropriate expressions for the four displacement quantities, we get

$$\begin{aligned}
\frac{a^2 B_n}{D(1-\nu)} + \frac{1}{\sqrt{D}} [C_n \text{ber}_n(\lambda) + D_n \text{bei}_n(\lambda)] &= -P_0 AR^2 \\
\frac{na^2 B_n}{D(1-\nu)} + \frac{\lambda}{\sqrt{D}} [C_n \text{ber}'_n(\lambda) + D_n \text{bei}'_n(\lambda)] &= -nP_0 AR^2 \\
na(1+\nu) AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} - \frac{n(1+\nu)\sqrt{A}}{a} [C_n \text{bei}_n(\lambda) - D_n \text{ber}_n(\lambda)] &= \frac{n(1+\nu)}{4(n+1)} P_0 ARa
\end{aligned}$$

---

\* The particular solution given in Ref. 5 for the radial displacement component is actually the expression for the meridional displacement component.

$$\begin{aligned}
& -na(1+\nu) AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} + \frac{\lambda(1+\nu)\sqrt{A}}{a} [C_n \text{bei}'_n(\lambda) - D_n \text{ber}'_n(\lambda)] \\
& = -\frac{(n+2)(1+\nu)}{4(n+1)} P_0 ARa \quad .
\end{aligned} \tag{VII-5}$$

Solving (VII-5) for  $A_n$  and  $B_n$ , we have

$$A_n = \frac{P_0 R}{8n(n+1)} [(n+2) Y_1 + nY_2 - 8(n+1)(Y_3 - nY_4)] \quad , \tag{VII-6}$$

$$B_n = -\frac{P_0 a^2 (1-\nu^2)}{(2\mu)^4} [Y_4 - 4(n+1)(Y_6 - nY_7)] \quad , \tag{VII-7}$$

where the  $Y_i$ 's are just those given by (IV-5). If  $\mu \gg n \geq 1$ , we have

$$A_n \sim \frac{P_0 R}{4n} \left[ 1 - \frac{3-\nu}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \quad , \tag{VII-8}$$

$$B_n \sim -\frac{P_0 a^2 (1-\nu^2)}{(2\mu)^4} \left[ 1 - \frac{(n+1)(3-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \quad . \tag{VII-9}$$

If terms of the order  $1/\mu$  are also suppressed, (VII-8) and (VII-9) become

$$A_n \sim \frac{P_0 R}{4n} [1 + O(\frac{1}{\mu})] \quad , \tag{VII-10}$$

$$B_n \sim -\frac{P_0 a^2 (1-\nu^2)}{(2\mu)^4} [1 + O(\frac{1}{\mu})] \quad . \tag{VII-11}$$

For  $\mu \gg n \geq 1$ , we have

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{a^2}{Rh} \frac{1}{\mu^4}\right) \quad . \tag{VII-12}$$

If the shell is isotropic and homogeneous,  $a^2/Rh = O(\mu^2)$ , and therefore

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{1}{\mu^2}\right) \quad . \tag{VII-13}$$

Thus the interior of the shell is in a membrane stress state.

### C. Simply Supported Edge

If the shell is simply supported, the conditions (VII-4) are replaced by

$$u = v = w = M_r = 0 \quad (\text{at } \rho = 1) \quad . \tag{VII-14}$$

Substituting into (VII-14) the appropriate expressions for the displacement components and the bending moment for the present problem, we have

$$\frac{a^2 B_n}{D(1-\nu)} + \frac{1}{\sqrt{D}} [C_n \text{ber}_n(\lambda) + D_n \text{bei}_n(\lambda)] = -P_0 AR^2$$

$$\begin{aligned}
n(n-1) B_n - \frac{1}{R\sqrt{A}} [C_n g_{rc}(\lambda) - D_n g_{rd}(\lambda)] &= - \frac{n(n-1)(1-\nu) P_0 a^2}{\lambda^4} \\
na(1+\nu) AA_n + \frac{\alpha a^2 B_n}{D(n+1)(1-\nu)} - \frac{n(1+\nu)\sqrt{A}}{a} [C_n \text{bei}_n(\lambda) - D_n \text{bei}_n(\lambda)] &= \frac{n(1+\nu) P_0 ARa}{4(n+1)} \\
-na(1+\nu) AA_n + \frac{\alpha a^2 B_n}{D(n+1)(1-\nu)} + \frac{\lambda(1+\nu)\sqrt{A}}{a} [C_n \text{bei}'_n(\lambda) - D_n \text{bei}'_n(\lambda)] \\
&= - \frac{(n+2)(1+\nu) P_0 ARa}{4(n+1)} \quad . \quad (VII-15)
\end{aligned}$$

Solving (VII-15) for  $A_n$  and  $B_n$ , we get

$$A_n = \frac{P_0 R}{8n(n+1) \Delta_6} \left[ nK_1 + (n+2) K_2 - \frac{4(n+1)}{\lambda} K_3 + \frac{4n(n^2-1)(1-\nu)}{\lambda^3} K_4 \right] \quad , \quad (VII-16)$$

$$B_n = - \frac{P_0 a^2 (1-\nu^2)}{(2\mu)^4 \Delta_6} (K_5) \quad , \quad (VII-17)$$

where

$$K_1 = \alpha_4 - \left[ \frac{(1-\nu) + (n+1)(1+\nu)}{\lambda} \right] + \frac{(n+1)(1-\nu^2)}{\lambda^2} \left( \alpha_2 - \frac{n}{\lambda} \alpha_3 \right)$$

$$K_2 = \alpha_4 - \frac{1-\nu}{\lambda} - \frac{n(n+1)(1-\nu^2)}{\lambda^3} \left( \alpha_3 - \frac{n}{\lambda} \alpha_4 \right)$$

$$K_3 = \left[ 1 - \frac{n^2(n^2-1)(1-\nu^2)}{\lambda^4} \right] - \frac{1-\nu}{\lambda} \left[ \alpha_2 - \frac{n(n-1)}{\lambda} \alpha_3 - \frac{n^3}{\lambda^2} \alpha_4 \right]$$

$$K_4 = \alpha_3 + \frac{n}{\lambda} \alpha_4 - \frac{n(n+1)(1+\nu)}{\lambda^2}$$

$$K_5 = \alpha_4 \left[ 1 + \frac{2n^2(n+1)(1-\nu)}{\lambda^4} \right] - \frac{1}{\lambda} [(1-\nu) + 2(n+1)] + \frac{2(n+1)(1-\nu)}{\lambda^2} \left[ \alpha_2 - \frac{2n}{\lambda} \alpha_3 \right]$$

$$\begin{aligned}
\Delta_6 &= \alpha_4 \left[ 1 + \frac{n^2(n+1)(1-\nu^2)}{2\lambda^4} \right] - \frac{1}{\lambda} [(1-\nu) + \frac{(n+1)(1+\nu)}{2}] \\
&\quad + \frac{(n+1)(1-\nu^2)}{2\lambda^2} \left[ \alpha_2 - \frac{2n}{\lambda} \alpha_3 \right] \quad .
\end{aligned}$$

The  $\alpha_i$ 's are defined by (III-7).

For  $\mu \gg n \geq 1$ , we have the following asymptotic expressions for  $A_n$  and  $B_n$

$$A_n \sim \frac{P_0 R}{4n} \left[ 1 - \frac{(3-\nu)}{4\mu} + O\left(\frac{1}{\mu^2}\right) \right] \quad , \quad (VII-18)$$

$$B_n \sim - \frac{P_0 a^2 (1 - \nu^2)}{(2\mu)^4} \left[ 1 - \frac{(n+1)(3-\nu)}{4\mu} + O\left(\frac{1}{\mu}\right) \right] \quad (VII-19)$$

If terms of the order  $1/\mu$  are also neglected in comparison with unity, (VII-18) and (VII-19) become

$$A_n \sim \frac{P_0 R}{4n} \left[ 1 + O\left(\frac{1}{\mu}\right) \right] \quad (VII-20)$$

and

$$B_n \sim - \frac{P_0 a^2 (1 - \nu^2)}{(2\mu)^4} \left[ 1 + O\left(\frac{1}{\mu}\right) \right] \quad (VII-21)$$

A comparison of (VII-18) and (VII-19) with (VII-8) and (VII-9) suggests that, to the first approximation, the interior state of the shell is the same whether the edge of the shell is clamped or simply supported. To say it in another way: for shells with a small bending-to-stretching stiffness ratio so that  $\mu \gg n \geq 1$ , the interior of the shell is rather insensitive to the difference between the two types of edge supports. It is interesting that the second-order correction terms in (VII-18) and (VII-19) are exactly half the corresponding terms in (VII-8) and (VII-9).

#### D. Membrane Analysis

Since both types of edge support considered in the last two sections lead to a membrane interior state, we might suspect that the corresponding solution obtained by way of a membrane analysis would provide a good approximation to the exact solution for shells with a clamped or simply supported edge. We shall presently show that this is so.

Let us consider now a shell with vanishing bending stiffness. The governing equations for this membrane shell can be obtained from (II-1) by setting  $D = 0$ . Thus we have

$$\begin{aligned} \nabla^2 F_m &= -RP_n = -RP_0 \rho^n \cos n\theta \\ \nabla^2 w_m &= -RA \nabla^2 \nabla^2 F_m \end{aligned} \quad (VII-22)$$

The solutions to (VII-22) for  $F_m$  and  $w_m$  are (keeping in mind that the shell is closed at the apex and that the stress resultants and displacements are finite there)

$$\begin{aligned} F_m &= \left[ a^2 A_{mn} \rho^n - \frac{P_0 a^2 R}{4(n+1)} \rho^{n+2} \right] \cos n\theta \\ w_m &= \left[ \frac{a^2 B_{mn}}{D(1-\nu)} \rho^n + P_0 AR^2 \rho^n \right] \cos n\theta \end{aligned} \quad (VII-23)$$

Correspondingly, the tangential displacement components are

$$v_m = \left[ na(1+\nu) AA_{mn} \rho^{n-1} + \frac{a^3 B_{mn} \rho^{n+1}}{RD(n+1)(1-\nu)} - \frac{n(1+\nu) a P_0 AR \rho^{n+1}}{4(n+1)} \right] \sin n\theta \quad ,$$

$$u_m = \left[ -na(1 + \nu) AA_{mn}\rho^{n-1} + \frac{a^3 B_{mn}\rho^{n+1}}{RD(n+1)(1-\nu)} + \frac{(n+2)(1+\nu)aP_0AR\rho^{n+1}}{4(n+1)} \right] \cos n\theta \quad . \quad (\text{VII-24})$$

The constants  $A_{mn}$  and  $B_{mn}$  are determined by the usual membrane (tangentially) fixed edge conditions (Fig. 3).

$$u_m = v_m = 0 \quad (\text{at } \rho = 1) \quad . \quad (\text{VII-25})$$

The result is

$$A_{mn} = \frac{P_0 R}{4n} \quad , \quad B_{mn} = -\frac{P_0 a^2 (1-\nu^2)}{(2\mu)^4} \quad . \quad (\text{VII-26})$$

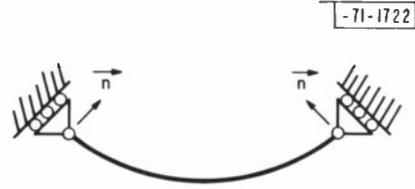


Fig. 3. Tangential support.

Thus we see that, except for the forces and moments associated with the inextensional bending of the shell (which are identically zero since the shell offers no resistance to bending), the membrane solution provides a good first approximation to the interior solution. If the portion of the above results associated with  $B_{mn}$  is regarded as the inextensional bending component of the complete solution so that the transverse shears and bending moments can be derived from the expression for  $w$  ( $D \neq 0$  for this purpose), a comparison of (VII-26) with (VII-10) and (VII-11) or with (VII-20) and (VII-21) shows that the result from such an analysis is, in fact, the leading term of the asymptotic expansion of the interior solution for shells with either a clamped edge or a simply supported edge.

### E. Shells Fixed Tangentially But Free Otherwise

In this section, we show that a membrane interior state is still ensured if the last two conditions in (VII-4) are replaced by  $R_r = M_r = 0$ . Moreover, if  $n = 1$ , we have the interesting result that the membrane solution turns out to be the exact solution of (II-1). Inconclusive though they may be, these results provide at least a partial and quantitative substantiation of the appropriateness of the edge support used in the membrane theory of shells.

Consider then a shell with vanishing tangential displacements and transverse-shear resultant and moment (Fig. 3).

$$u = v = M_r = R_r = 0 \quad (\text{at } \rho = 1) \quad . \quad (\text{VII-27})$$

For the present problem, (VII-27) takes the form

$$\begin{aligned}
& na(1+\nu) AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} - \frac{n(1+\nu)\sqrt{A}}{a} [C_n \text{bei}_n(\lambda) - D_n \text{ber}_n(\lambda)] = \frac{n(1+\nu)}{4(n+1)} P_0 ARa \\
& -na(1+\nu) AA_n + \frac{a^3 B_n}{RD(n+1)(1-\nu)} + \frac{\lambda(1+\nu)\sqrt{A}}{a} [C_n \text{bei}'_n(\lambda) - D_n \text{ber}'_n(\lambda)] \\
& = -\frac{(n+2)(1-\nu)}{4(n+1)} P_0 ARa \\
& -n(n-1) B_n + \frac{1}{R\sqrt{A}} [C_n g_{rc}(\lambda) - D_n g_{rd}(\lambda)] = n(n-1)(1-\nu) \frac{P_0 a^2}{\lambda^4} \\
& n^2(n-1) B_n + \frac{\lambda}{R\sqrt{A}} [C_n f_{nc}(\lambda) - D_n f_{nc}(\lambda)] = n^2(n-1)(1-\nu) \frac{P_0 a^2}{\lambda^4} . \tag{VII-28}
\end{aligned}$$

Solving (VII-28) for  $A_n$  and  $B_n$ , we get

$$A_n = \frac{P_0 R}{8n(n+1)} \left[ \frac{4n^2(n^2-1)(1-\nu)}{\lambda^4} (Z_1 + Z_2) + nZ_3 + (n+2)Z_4 \right] \tag{VII-29}$$

$$B_n = -\frac{P_0 a^2 (1-\nu^2)}{4\lambda^4} \left[ -\frac{2n^2(n^2-1)(1-\nu)}{\lambda^4} (Z_5 - Z_6) + Z_7 \right] \tag{VII-30}$$

where the  $Z_i$ 's were given by (V-5).

If  $\mu \gg n \geq 1$ , we have

$$A_n \sim \frac{P_0 R}{4n} \left[ 1 + O\left(\frac{1}{\mu^4}\right) \right] \tag{VII-31}$$

$$B_n \sim -\frac{P_0 a^2 (1-\nu^2)}{(2\mu)^4} \left[ 1 + O\left(\frac{1}{\mu^4}\right) \right] . \tag{VII-32}$$

For a homogeneous and isotropic shell (again  $\mu \gg n \geq 1$ ),

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{a^2}{Rh} \frac{1}{\mu^4}\right) = O\left(\frac{1}{\mu^2}\right) , \tag{VII-33}$$

which is the first part of our contention. Note that correction terms in (VII-31) and (VII-32) are  $O(1/\mu^4)$  while these same correction terms are  $O(1/\mu)$  in Secs. VII-B and VII-C.

For  $n = 1$ , (VII-29) and (VII-30) become [cf. (V-5)]

$$A_1 = \frac{P_0 R}{4} \tag{VII-34}$$

and

$$B_1 = \frac{-P_0 a^2 (1-\nu^2)}{(2\mu)^4} , \tag{VII-35}$$

respectively. Moreover, the last two conditions of (VII-28) become

$$\begin{aligned} C_1 g_{rc}(\lambda) - D_1 g_{rd}(\lambda) &= 0 \\ C_1 f_{nc}(\lambda) - D_1 f_{nd}(\lambda) &= 0 \end{aligned} \quad (\text{VII-36})$$

or

$$C_1 = D_1 = 0 \quad (\text{VII-37})$$

Equations (VII-34), (VII-35), and (VII-37) constitute the second part of our contention. It is not difficult to verify that the shell is still in a state of over-all static equilibrium.

### F. Shells Fixed Transversely But Free Otherwise

As a contrast to (VII-27), we consider next a shell supported in such a way that (Fig. 4)

$$N_r = N_{r\theta} = w = \frac{\partial w}{\partial r} = 0 \quad (\text{at } \rho = 1) \quad (\text{VII-38})$$

Observing (II-18), (VII-38) becomes

$$\begin{aligned} n(n-1) A_n + \frac{\lambda}{a^2 \sqrt{A}} [C_n f_{rc}(\lambda) - D_n f_{rd}(\lambda)] &= \frac{n-2}{4} P_0 R \\ n(n-1) A_n - \frac{n\lambda}{a^2 \sqrt{A}} [C_n f_{sc}(\lambda) - D_n f_{sd}(\lambda)] &= \frac{n}{4} P_0 R \\ \frac{a^2 B_n}{D(1-\nu)} + \frac{1}{\sqrt{D}} [C_n \text{ber}_n(\lambda) + D_n \text{bei}_n(\lambda)] &= -P_0 A R^2 \\ \frac{na^2 B_n}{D(1-\nu)} + \frac{\lambda}{\sqrt{D}} [C_n \text{ber}'_n(\lambda) + D_n \text{bei}'_n(\lambda)] &= -nP_0 A R^2 \end{aligned} \quad (\text{VII-39})$$

If  $n \geq 2$ , we may solve (VII-39) for  $A_n$  and  $B_n$  to get

$$A_n = \frac{P_0 R}{4(n+1) \Delta_8} \left[ \alpha_2 - \frac{2(n+1)}{\lambda} \alpha_3 + \frac{n(n+2)}{\lambda^2} \alpha_4 \right] \quad (\text{VII-40})$$

$$B_n = \frac{P_0 a^2 (1-\nu)}{2\lambda^3 (n+1) \Delta_8} \left[ 1 - \frac{2(n+1)}{\lambda} \left( \alpha_2 - \frac{2n\alpha_3}{\lambda} + \frac{n^2 \alpha_4}{\lambda^2} \right) \right] \quad (\text{VII-41})$$

with

$$\Delta_8 = \alpha_2 - \frac{2n}{\lambda} \alpha_3 + \frac{n^2}{\lambda^2} \alpha_4 \quad (\text{VII-42})$$

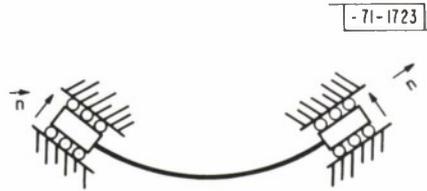


Fig. 4. Normal support.

The  $\alpha_1$ 's are defined by (III-7). For  $\mu \gg n > 1$ , we have

$$A_n \sim \frac{P_0 R}{4(n+1)} \left[ 1 - \frac{1}{\mu} + O\left(\frac{1}{\mu^2}\right) \right] \quad (\text{VII-43})$$

$$B_n \sim \frac{P_0 a^2 (1-\nu)}{(n+1)(2\mu)^3} \left[ 1 - \frac{2n+3}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \quad (\text{VII-44})$$

For  $n = 1$ , (VII-39) becomes

$$\begin{aligned} & -\frac{\lambda}{a^2 \sqrt{A}} \{C_1 [\text{bei}'_1(\lambda) - \frac{1}{\lambda} \text{bei}_1(\lambda)] - D_1 [\text{ber}'_1(\lambda) - \frac{1}{\lambda} \text{ber}_1(\lambda)]\} = -\frac{P_0 R}{4} \\ & -\frac{\lambda}{a^2 \sqrt{A}} \{C_1 [\text{bei}'_1(\lambda) - \frac{1}{\lambda} \text{bei}_1(\lambda)] - D_1 [\text{ber}'_1(\lambda) - \frac{1}{\lambda} \text{ber}_1(\lambda)]\} = \frac{1}{4} P_0 R \\ & \frac{a^2 B_1}{D(1-\nu)} + \frac{1}{\sqrt{D}} [C_1 \text{ber}_1(\lambda) + D_1 \text{bei}_1(\lambda)] = -P_0 A R^2 \\ & \frac{a^2 B_1}{D(1-\nu)} + \frac{\lambda}{\sqrt{D}} [C_1 \text{ber}'_1(\lambda) + D_1 \text{bei}'_1(\lambda)] = -P_0 A R^2 \quad (\text{VII-45}) \end{aligned}$$

The first two conditions of (VII-45) are the same; we are then left with three relations for the three unknown  $B_1$ ,  $C_1$  and  $D_1$  appearing in these relations. The solution for  $B_1$  is again given by (VII-41) with  $n = 1$ .

With all the prescribed edge constraints satisfied, we are still left with an arbitrary constant  $A_1$  which appears only in the expressions for the tangential displacement components  $u$  and  $v$ . A closer examination, however, reveals that the contribution of the terms associated with  $A_1$  to  $u$  and  $v$  is exclusively in the nature of rigid body motion.

In all cases, as long as  $\mu \gg n \geq 1$ , we have, for a homogeneous and isotropic shell,

$$\frac{\sigma_B}{\sigma_D} = O\left(\frac{a^2}{R h} \frac{1}{\mu^3}\right) = O\left(\frac{1}{\mu}\right) \quad (\text{VII-46})$$

Although the membrane stresses again dominate in the interior, the relative importance of the two stress states as given by (VII-46) is more marginal than that exhibited by the shells previously considered in this section.

### G. Free Edge

Since several types of edge constraints have all led to a membrane interior state for the axial surface load under consideration, we would naturally ask whether there is any set of edge constraints which would lead to an inextensional bending interior state. To answer this, let us consider a shell with a free edge so that

$$N_r = N_{r\theta} = M_r = R_r = 0 \quad (\text{at } \rho = 1) \quad (\text{VII-47})$$

Clearly, this is a permissible set of edge conditions only if the applied load is self-equilibrating. Therefore, we must restrict the integer  $n$  to greater than unity. Observing (II-18) and (VII-3), (VII-47) becomes

$$n(n-1) A_n + \frac{\lambda}{a^2 \sqrt{A}} [C_n f_{rc}(\lambda) - D_n f_{rd}(\lambda)] = \frac{n-2}{4} P_0 R$$

$$\begin{aligned}
n(n-1) A_n - \frac{n\lambda}{a^2 \sqrt{A}} [C_{n\text{fsc}}(\lambda) - D_{n\text{fsc}}(\lambda)] &= \frac{n}{4} P_0 R \\
-n(n-1) B_n + \frac{1}{R \sqrt{A}} [C_{n\text{grc}}(\lambda) - D_{n\text{grd}}(\lambda)] &= n(n-1) (1-\nu) \frac{P_0 a^2}{\lambda^4} \\
n^2(n-1) B_n + \frac{1}{R \sqrt{A}} [C_{n\text{fnc}}(\lambda) - D_{n\text{fnd}}(\lambda)] &= -n^2(n-1) (1-\nu) \frac{P_0 a^2}{\lambda^4} .
\end{aligned} \tag{VII-48}$$

Solving (VII-48) for  $A_n$  and  $B_n$ , we get

$$A_n = \frac{P_0 R}{8n(n-1)} [nX_1 + (n-2) X_2] \tag{VII-49}$$

$$B_n = \frac{P_0 a^2}{4n^2(n^2-1)} \left[ X_4 - \frac{4n^2(n^2-1)(1-\nu)}{\lambda^4} (X_1 + X_2) \right] , \tag{VII-50}$$

where the  $X_i$ 's are just those given by (III-5). For  $\mu \gg n > 1$ , we have

$$A_n \sim \frac{P_0 R}{4n} \left[ 1 + \frac{(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] \tag{VII-51}$$

$$B_n \sim \frac{P_0 a^2}{4n^2(n^2-1)} \left[ 1 - \frac{(n-1)(1-\nu)}{2\mu} + O\left(\frac{1}{\mu^2}\right) \right] . \tag{VII-52}$$

If, in addition, the shell is homogeneous and isotropic, we have

$$\frac{\sigma_D}{\sigma_B} = O\left(\frac{Rh}{a^2}\right) = O\left(\frac{1}{\mu^2}\right) . \tag{VII-53}$$

Thus the interior of the shell is primarily in a state of inextensional bending.

### VIII. SUMMARY AND REMARKS

In the foregoing, we have solved a series of boundary value problems exactly and investigated the corresponding asymptotic behavior of the so-called interior solutions subject to the assumption  $\mu \gg n$ . For shells without surface loads, the present work supplemented known results,<sup>4,5</sup> with the explicit determination of the second-order corrections to the leading term of the asymptotic interior solutions. These correction terms enabled us to establish the conditions under which the leading terms provide an adequate first approximation to the exact interior solutions. They also led us to some refinements of the correspondence between the interior stress state and the boundary conditions. Exact and asymptotic influence coefficients were obtained. For shells with polar harmonic axial surface loads, our results showed that the shell interior is in a membrane state for the various types of edge support considered. On the other hand, if the edge is free, the interior of the shell is in a state of inextensional bending. It is hoped that these results may contribute to a better understanding of the interplay between the interior membrane and inextensional bending stresses.

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<p>The present work is concerned with the nature of the interior solution and the influence coefficients of shallow, spherical, thin, elastic shells (or equivalently a shallow, thin, elastic paraboloidal shell of revolution) which are homogeneous, isotropic, closed at the apex, and of uniform thickness. The investigation is carried out within the framework of the usual shallow shell theory for small displacements and negligible transverse-shear deformations. Exact interior solutions are obtained for shells acted upon by edge loads and edge moments. The constants of integration associated with these interior solutions are expanded asymptotically in inverse powers of a large parameter. Retaining only the leading term of these expansions leads (in most cases) to known approximate results. Explicit expressions for the second-order terms are obtained. It is shown that these second-order terms play a significant role in a certain class of problems. The relative importance of the membrane and inextensional bending stresses in the interior of the shell is discussed. The exact and asymptotic influence coefficients are obtained. The interior stress state of shells subjected to polar harmonic axial surface loads is also investigated by the same procedure.</p>			

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